Solutions

Foundation

1.

3. Since f and g are inverse of each other, we have $f(g(x)) = x$

$$f'(g(x)) \cdot g'(x) = 1$$

$$\implies f'(g(1)) \cdot g'(1) = 1$$

Notice that $f(0) = 1 \implies g(1) = 0$. Returning to the original equation

$$f'(0) \cdot g'(1) = 1.$$

Since $f(x) = x^3 + e^{\frac{x}{2}} \implies f'(x) = 3x^2 + \frac{1}{2}e^{\frac{x}{2}} \implies f'(0) = \frac{1}{2}$. So, $g'(1) = 2$.

5. From the given functional equations we have,

$$f(3000) = 3001,\ f(2999) = 3000,\ f(2998) = 2999, \text{ and similarly}$$

$$f(2997) = 3001, f(2996) = 3000, f(2995) = 2999.$$

Claim $f(3k) = 3001$, $f(3k-1) = 3000$, $f(3k-2) = 2999$ for $3k \leq 2997$.

We have $f(3k-3) = f(f(3k+2)) = f(3000) = 3001$. Similarly, you can prove it for others. Therefore, $f(2022) = 3001$.

6.(a) Claim $f(2n+1) = 2$ and $f(2n) = 1$ for every non-negative integer n. Use strong induction to prove this claim.

(b) Say that $f(0) = 1$, then $f(2 \cdot 0) = f(f(0)) \implies f(1) = f(0)$, but using the second functional equation $f(1) = f(0) + 1$. This is a contradiction.

(c) Say that $f(0) = 2^k$ for some non-negative integer k. Since we have $f(2n) = f(f(n))$. So, we have $f(2 \cdot 0) = f(f(0)) = f(2^k) \implies f(0) = f(2^k) = 2^k$. Since $f(2n+1) = f(2n) + 1 \implies f(1) = f(0) + 1 = 2^k + 1$. Now, $f(2^k + 1) = f(2^k) + 1 = 2^k + 1$.

So, for any non-negative integer l, $f(2^l) = \underbrace{f(f(f(\ldots f(1))))}_{l+1 \text{ times}} = 2^k + 1$.

However, $f(2^l) = \underbrace{f(f(f(\ldots f(2))))}_{l \text{ times}} = 2^k$, which is a contradiction.

7. (a) Say for all values of x, $f(x) \neq x$ and yet $f(f(x)) = f(x)$. Let that value of x be x_0. Say that $f(x_0) = y_0$ for $x_0 \neq y_0$ and $f(f(x_0)) = f(x_0) = y_0$. Since $f(x_0) = y_0 \implies f(f(x_0)) = f(y_0) = f(x_0) = y_0$. This is a contradiction to $f(x) \neq x$ for all x.

(b) Now that we have proved in part (a) that for $f(f(x)) = f(x)$, there must be at least one x for which $f(x) = x$. So, say that $f(x) = x$ for k elements of the set $\{1,2,3,\ldots,n\}$. These k elements can be chosen from n elements in $\binom{n}{k}$ ways. The remaining $(n-k)$ elements can be mapped to those k elements in k^{n-k} ways. Therefore, the total number of such functions is equal to:

$$\sum_{k=1}^{n} \binom{n}{k} k^{n-k}.$$

8. If m is a perfect square, then we are obviously done. Hence, assume that $m = k^2 + j$ for $1 \leq j \leq 2m$. Hence $f(m) = k^2 + k + j$. Consider the following two sets

$$A = \{m \,\epsilon\, \mathbb{N} : m = k^2 + j,\ 0 \leq j \leq k.\}$$

$$B = \{m \,\epsilon\, \mathbb{N} : m = k^2 + j,\ k+1 \leq j \leq 2k.\}$$

Suppose that m is in the set B. Then

$$f(m) = k^2 + j + k = (k+1)^2 + (j - (k+1))$$

So either $f(m)$ is in the set A or it is a perfect square. If it is a perfect square, we are done, so let us assume that it lies in the set A. In that case, $k^2 < f(m) < (k+1)^2 \implies \lfloor f(m) \rfloor = k$.

$$\implies f^2(m) = (k+1)^2 + (j-1)$$

$$\implies f^{2j}(m) = (k+j)^2.$$

10. We are looking for total number of solutions of the equation

$$y_1 \leq f(x_1) \leq f(x_2) \leq \cdots \leq f(x_{50}) \leq y_{20}.$$

This problem is equivalent to distributing 50 identical objects among 20 people so that each person gets at least one object. This can be done in $\binom{49}{19}$ ways.

12. The inequality can be rearranged as

$$(1^2 + 1^2)\left(\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2\right) \geq \frac{25}{2}$$

This is a clear invocation of Cauchy-Schwarz

$$(1^2+1^2)\left(\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\right)\geq\left(1\cdot\left(a+\frac{1}{a}\right)+1\cdot\left(b+\frac{1}{b}\right)\right)^2$$

$$\implies (1^2+1^2)\left(\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\right)\geq\left(a+b+\frac{1}{a}+\frac{1}{b}\right)^2$$

$$2\left(\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\right)\geq\left(1+\frac{1}{a}+\frac{1}{b}\right)^2 \tag{A}$$

Using AM-HM inequality, we have: $\frac{a+b}{2}\geq\frac{2}{\frac{1}{a}+\frac{1}{b}} \implies \frac{1}{a}+\frac{1}{b}\geq 4$. Going back to (A).

$$2\left(\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\right)\geq(1+4)^2=25.$$

Limits

4. Say $1 - x = t$, so the given problem becomes

$$\lim_{t\to 0} t\tan(1-t)\frac{\pi}{2} = \lim_{t\to 0} t\tan\left(\frac{\pi}{2} - \frac{\pi}{2}t\right)$$

$$\begin{aligned}
\lim_{t\to 0} t\cot\left(\frac{\pi}{2}t\right) &= \lim_{t\to 0}\frac{t}{\tan\left(\frac{\pi}{2}t\right)} \\
&= \lim_{t\to 0}\left[\frac{1}{\frac{\tan\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}}\cdot\frac{\pi}{2}\right] \\
&= \frac{\pi}{2}\cdot\lim_{t\to 0}\frac{1}{\frac{\tan\left(\frac{\pi}{2}t\right)}{\frac{\pi}{2}t}} \\
&= \frac{\pi}{2}
\end{aligned}$$

5. Let $f(x) = \cos 2x\cdot\cos 4x\cdot\cos 6x\cdot\cos 8x\cdot\cos 10x$, then compute the limit

$$\lim_{x\to 0}\frac{(1-(f(x))^3)}{55\sin^2 x}$$

6. Consider the sum, $\sum_{r=0}^{n-1}\cos\left(\frac{r\pi}{2n}\right)$

$$\sum_{r=0}^{n-1}\cos\left(\frac{r\pi}{2n}\right) = \sum_{r=0}^{n-1} Re(e^{\frac{ir\pi}{2n}})$$

Where $Re(Z)$ refers to the real part of complex number Z.

$$\sum_{r=0}^{n-1} Re(e^{\frac{ir\pi}{2n}}) = Re\left(\sum_{r=0}^{n-1} e^{\frac{ir\pi}{2n}}\right) = \frac{1}{2}\left(1+\cot\frac{4\pi}{n}\right).$$

$$\begin{aligned}
\implies \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\cos\left(\frac{r\pi}{2n}\right) &= \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{2}\left(1+\cot\frac{4\pi}{n}\right). \\
&= \frac{1}{2}\cdot\lim_{n\to\infty}\left(\frac{1}{n}+\frac{\cot\frac{4\pi}{n}}{\frac{1}{n}}\right) = \frac{2}{\pi}.
\end{aligned}$$

Sum of Cosines in Arithmetic Progression

$$\sum_{k=0}^{n-1}\cos(\alpha+k\beta) = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\cos\left(\alpha+\frac{(n-1)\beta}{2}\right)$$

7. Notice that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{1}{n}}=e.$

$$\text{The given limit is} : \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{\frac{8n^3}{\pi}\sin\left(\frac{\pi}{2n}\right)} = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2\cdot\frac{8n^3}{n^2\pi}\sin\left(\frac{\pi}{2n}\right)}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2\cdot\frac{8n}{\pi}\sin\left(\frac{\pi}{2n}\right)} = \lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2\cdot 4\cdot\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}}}$$

$$\text{Notice, } \lim_{n\to\infty} = \left(1+\frac{1}{n^2}\right)^{n^2} = e, \text{ and } \lim_{n\to\infty}\frac{\sin\frac{\pi}{2n}}{\frac{\pi}{2n}} = 1.$$

So, the given limit is e^4.

10. We know that $(1+x)^{2n} = \sum_{r=0}^{2n}\binom{2n}{r}x^r$ and $(1-x)^{2n} = \sum_{r=0}^{2n}\binom{2n}{r}(-x)^r$

$$\implies 2\sum_{r=0}^{n}\binom{2n}{r}x^r = (1+x)^n + (1-x)^n$$

$$\text{Or, } \sum_{r=0}^{n}\binom{2n}{2r}x^{2r} = \frac{(1+x)^n+(1-x)^n}{2}$$

$$\text{Or, } \sum_{r=0}^{n}\binom{2n}{2r}x^{r} = \frac{(1+\sqrt{x})^n+(1-\sqrt{x})^n}{2}.$$

$$\text{Similarly, } \sum_{r=0}^{n-1}\binom{2n}{2r+1}x^{r} = \frac{(1+\sqrt{x})^n-(1-\sqrt{x})^n}{2x}.$$

$$\lim_{n\to\infty}\frac{\sum\limits_{r=0}^{n}\binom{2n}{2r}\cdot 3^r}{\sum\limits_{r=0}^{n-1}\binom{2n}{2r+1}\cdot 3^r} = \lim_{n\to\infty}\frac{\frac{(1+\sqrt{3})^n+(1-\sqrt{3})^n}{2}}{\frac{(1+\sqrt{3})^n-(1-\sqrt{3})^n}{2\sqrt{3}}}$$

$$= \sqrt{3}\lim_{n\to\infty}\frac{(1+\sqrt{3})^n+(1-\sqrt{3})^n}{(1+\sqrt{3})^n-(1-\sqrt{3})^n}$$

$$= \sqrt{3}\lim_{n\to\infty}\frac{1+\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)^n}{1+\left(\frac{1-\sqrt{3}}{1+\sqrt{3}}\right)^n} = \sqrt{3}.$$

11.

$$\tan^{-1}\left(\frac{1}{1+m+m^2}\right) = \tan^{-1}\left(\frac{(m+1)-(m)}{1+m(m+1)}\right) = \tan^{-1}(m+1)-\tan^{-1}(m).$$

It is a Telescopic sum and now proceed.

14. The limit to be evaluated is:

$$\lim_{n\to\infty} \frac{(1+\alpha)^n + (\alpha-1)^n}{(2\alpha)^n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2\alpha}\right)^n + \left(\frac{1}{2} - \frac{1}{2\alpha}\right)^n.$$

$$\text{Now the given limit is equal to} = \begin{cases} 1, & \text{if } \alpha = 1 \\ 0, & \text{if } \alpha > 1 \\ \infty, & \text{if } 0 < \alpha < 1. \end{cases}$$

18. (a) Since $0 < a_1 < 1$, it can be inductively proved that all subsequent terms of the sequence are positive. And we have

$$a_{n+1} = \frac{3a_n}{2+a_n} \implies a_{n+1} - a_n = \frac{a_n - a_n^2}{2+a_n} > 0, \text{ since } 0 < a_1 < 1.$$

So, the sequence is increasing.

Claim: The sequence, $\{a_n\}_{n\geq 1}$, is bounded by 1 from above. Proof using induction: Clearly, this is true for $n = 1$. Assume that it is true for $n = k$. And let us prove that this is true for $n = k+1$.

$$a_{k+1} = \frac{3a_k}{2+a_k} \implies a_{k+1} - 1 = \frac{2(a_k-1)}{2+a_k} < 0 \implies a_{k+1} < 1.$$

The sequence is monotonous and bounded, therefore by monotone convergence theorem the sequence will be convergent. Say $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$.

$$\implies L = \frac{3L}{2+L} \implies L^2 + 2L = 3L \implies L = 0 \text{ or } 1.$$

Since the sequence was increasing and $0 < a_1 < 1 \implies L = 1$. Similarly, work the other part.

19. Consider the problem $\sqrt{2+\sqrt{2+\sqrt{2+\ldots\sqrt{2}}}}$ having $(n-1)$ radical signs. Since $\sqrt{2} = 2\cos\frac{\pi}{4} \implies \sqrt{2+\sqrt{2}} = \sqrt{2+2\cos\frac{\pi}{4}} = \sqrt{2(1+\cos\frac{\pi}{4})} = \sqrt{2\cdot 2\cos^2\frac{\pi}{8}} = 2\cos\frac{\pi}{8}$. So, inductively one can prove that $\sqrt{2+\sqrt{2+\sqrt{2+\ldots\sqrt{2}}}}$ with $(n-1)$ radical signs is $2\cos\frac{\pi}{2^n}$. And therefore the original problem, which is $\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}$ with n radical signs becomes $2 - 2\cos\frac{\pi}{2^n} =$

$2(1-\cos\frac{\pi}{2^n}) = 4\cdot\sin^2\frac{\pi}{2^{n+1}}$. Therefore, the given expression with n radical signs is equal to $2\sin\frac{\pi}{2^{n+1}}$.

$$\therefore \lim_{n\to\infty} 2^n A_n = \lim_{n\to\infty} 2^{n+1}\sin\left(\frac{\pi}{2^{n+1}}\right) = \pi \lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{2^{n+1}}\right)}{\frac{\pi}{2^{n+1}}} = \pi.$$

20. Consider $(3+2\sqrt{2})^n = I + f$, where $I = \lfloor (3+2\sqrt{2})^n \rfloor$ and $0 < f < 1$. Now $f = \{x\}$, the fractional part of x. Consider $f' = (3-2\sqrt{2})^n$. Obviously $0 < f' < 1$ and $(I+f)\cdot f' = 1$.

$$I + f + f' = (3+2\sqrt{2})^n + (3-2\sqrt{2})^n = 2\left(3^n + \binom{n}{2}3^{n-2}\cdot 8 + \dots\right)$$

Which is an integer. Since I is also an integer, it forces $f + f'$ to be an integer. But $0 < f + f' < 2 \implies f + f' = 1 \implies f = 1 - f'$.

$$\lim_{n\to\infty} f = 1 - \lim_{n\to\infty} f'.$$

$$\implies \lim_{n\to\infty} f = 1 - \lim_{n\to\infty}\{(3-2\sqrt{2})^n\} = 1.$$

22.

$$a_n = \int_0^n f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \cdots + \int_{n-1}^n f(x)\,dx.$$

Consider $\int_I^{I+1} f(x)\,dx$, where I is an integer. Define $x - I = y \implies dx = dy$.

$$\int_I^{I+1} f(x)\,dx = \int_0^1 f(y+I)\,dy.$$

Since $f(x+1) = \frac{1}{2}f(x) \implies f(x+2) = \frac{1}{2}f(x+1) = \frac{1}{2^2}f(x)$. So inductively, we can argue $f(x+I) = \frac{1}{2^I}f(x)$.

$$a_n = \int_0^n f(x)\,dx = \sum_{I=0}^{n-1}\int_I^{I+1} f(x+I)\,dx = \sum_{I=0}^{n-1}\int_0^1 \frac{1}{2^I}f(x)\,dx$$

$$\implies a_n = \left(\int_0^1 f(x)\,dx\right)\left(\sum_{I=0}^{n-1}\frac{1}{2^I}\right)$$

$$\implies \lim_{n\to\infty} a_n = \left(\int_0^1 f(x)\,dx\right)\lim_{n\to\infty}\left(\sum_{I=0}^{n-1}\frac{1}{2^I}\right) = 2\int_0^1 f(x)\,dx.$$

23. The probability of exactly one handshake is equal to $\binom{n}{2}\cdot\frac{2}{n^2}\cdot\left(1-\frac{2}{n^2}\right)^{\binom{n}{2}-1}$.

The probability of no handshakes is equal to $\left(1-\frac{2}{n^2}\right)^{\binom{n}{2}}$.

$$p_n = \binom{n}{2}\cdot\frac{2}{n^2}\cdot\left(1-\frac{2}{n^2}\right)^{\binom{n}{2}-1} + \left(1-\frac{2}{n^2}\right)^{\binom{n}{2}}$$

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty}\left[\binom{n}{2}\cdot\frac{2}{n^2}\cdot\left(1-\frac{2}{n^2}\right)^{\binom{n}{2}-1} + \left(1-\frac{2}{n^2}\right)^{\binom{n}{2}}\right]$$

Consider the limit $\lim\limits_{n\to\infty}\left(1-\frac{2}{n^2}\right)^{\binom{n}{2}} = \lim\limits_{n\to\infty}\left(1-\frac{2}{n^2}\right)^{\frac{n(n-1)}{2}}$

$$= \lim_{n\to\infty}\left[\left(1-\frac{2}{n^2}\right)^{\frac{n^2}{2}}\right]^{\frac{n(n-1)}{n^2}} = \frac{1}{e}.$$

Similarly, the other limit is also $\frac{1}{e}$, hence $\lim\limits_{n\to\infty} p_n = \frac{2}{e}$.

24. The given limit $\lim\limits_{n\to\infty} x_n(\log n)^t$ can be rewritten as $\lim\limits_{n\to\infty}(nx_n)\cdot\frac{(\log n)^t}{n}$. Now the limit $\lim\limits_{n\to\infty}\frac{(\log n)^t}{n}$ is zero for all real values of t, which can be seen by repeatedly applying the L'Hôpital's rule. Therefore, the entire limit is zero for all real values of t.

25. Say the number $8^n = (a_1a_2\ldots a_m)_6$. That is, when 8^n is written in base 6 it contains $m = p(n)$ digits. Clearly

$$6^{m-1} \le 8^n \le 6^m - 1 \le 6^m$$
$$\implies (m-1)\log 6 \le n\log 8 \le m\log 6$$
$$\implies \frac{m-1}{n}\log 6 \le \log 8 \le \frac{m}{n}\log 6$$
$$\implies \lim_{n\to\infty}\frac{m-1}{n}\log 6 \le \lim_{n\to\infty}\log 8 \le \lim_{n\to\infty}\frac{m}{n}\log 6$$
$$\implies \lim_{n\to\infty}\frac{m}{n} \ge \frac{\log 8}{\log 6}, \text{ and}, \lim_{n\to\infty}\frac{m}{n} \le \frac{\log 8}{\log 6}$$

The given limit $\lim\limits_{n\to\infty}\frac{m}{n} = \lim\limits_{n\to\infty}\frac{p(n)}{n}$ has been sandwiched between two equal numbers and therefore must be equal to this number $\implies \lim\limits_{n\to\infty}\frac{p(n)}{n} = \frac{\log 8}{\log 6}$. Similarly, once can argue $\lim\limits_{n\to\infty}\frac{q(n)}{n} = \frac{\log 6}{\log 4}$.

$$\lim_{n\to\infty}\frac{p(n)q(n)}{n^2} = \frac{\log 8}{\log 4} = \frac{3}{2}.$$

28. $\lim\limits_{n\to\infty}\binom{n}{k}\left(\frac{\mu}{n}\right)^k\left(1-\frac{\mu}{n}\right)^{n-k} = \lim\limits_{n\to\infty}\binom{n}{k}\left(\frac{\mu}{n}\right)^k\left(1-\frac{\mu}{n}\right)^n\frac{1}{(1-\frac{\mu}{n})^k}$.

$$= \lim_{n\to\infty}\frac{1}{k!}\cdot\frac{n(n-1)(n-2)\cdots(n-(k-1))}{n^k}\cdot\mu^k\cdot\left(1-\frac{\mu}{n}\right)^n\frac{1}{(1-\frac{\mu}{n})^k}$$

$$= \lim_{n\to\infty} \frac{1}{k!} \cdot \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-(k-1)}{n} \cdot \mu^k \cdot \left(1-\frac{\mu}{n}\right)^n \frac{1}{(1-\frac{\mu}{n})^k}$$

$$= \lim_{n\to\infty} \frac{1}{k!} \cdot \frac{n}{n} \cdot \left(1-\frac{1}{n}\right) \cdot \left(1-\frac{2}{n}\right) \cdots \left(1-\frac{k-1}{n}\right) \cdot \mu^k \cdot \left(1-\frac{\mu}{n}\right)^n \frac{1}{(1-\frac{\mu}{n})^k}$$

Notice $\lim_{n\to\infty} \left(1-\frac{r}{n}\right) = 1$ and $\lim_{n\to\infty} \left(1-\frac{\mu}{n}\right)^n = e^{-\mu}$ and $\lim_{n\to\infty} \frac{1}{(1-\frac{\mu}{n})^k} = 1$. So, the given limit is equal to $\frac{1}{k!}\mu^k e^{-\mu}$.

This is Poisson's approximation of Binomial distribution as n becomes very large.

29. Refer to problem number 47 in the chapter Integrals where we estimate the Harmonic sum and we refer to the same diagram. It can be seen that

$$\log(n+1) < \frac{1}{1} + \frac{1}{2} + \ldots \frac{1}{n} < 1 + \log n$$

It can be seen that $\frac{1}{1} + \frac{1}{2} + \ldots \frac{1}{n} - \log n < 1$. Now between k and $k+1$ it can be seen that the difference between the area bounded by the rectangle and the area bounded by the curve is

$$\frac{1}{k} - \log\left(\frac{k+1}{k}\right)$$

$$= \int_0^{\frac{1}{n}} \frac{t}{1+t}\, dt \geq \int_0^{\frac{1}{n}} \frac{t}{1+\frac{1}{n}}\, dt = \frac{1}{2n(n+1)}$$

$$\implies \sum_{k=1}^{n} \left(\frac{1}{k} - \log\left(\frac{k+1}{k}\right)\right) \geq \sum_{k=1}^{n} \frac{1}{2n(n+1)}.$$

Notice that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n(n+1)} = \frac{1}{2}.$$

30. 'Refer to problem number 47 in the chapter. We have established the bounds that

$$\log(n+1) < \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \log n \tag{A}$$

So, the given problem is equivalent to

$$e^{\lim\limits_{n\to\infty} \left(\frac{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}}{n^2}\right)\cdot n}$$

$$= e^{\lim\limits_{n\to\infty} \left(\frac{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}}{n}\right)}$$

Using (A), we can say that

$$\frac{\log(n+1)}{n} < \frac{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}}{n} < \frac{1+\log n}{n}$$

$$\implies \lim_{n\to\infty}\frac{\log(n+1)}{n} \le \lim_{n\to\infty}\frac{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}}{n} \le \lim_{n\to\infty}\frac{1+\log n}{n}$$

But, $\displaystyle\lim_{n\to\infty}\frac{\log(n+1)}{n} = \lim_{n\to\infty}\frac{1+\log n}{n} = 0 \implies \lim_{n\to\infty}\frac{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0.$

Therefore, the given limit is 1.

31. Say that $mt > 1$. Then there must be some p for which $2^p \le m < 2^{p+1}$. Since f is increasing, therefore $f(2^p t) \le f(mt) < f(2^{p+1}t)$.

$$\implies \frac{f(2^p t)}{f(2^p t)} \le \frac{f(mt)}{f(2^p t)} < \frac{f(2^{p+1}t)}{f(2^p t)}, \text{ Since f is positive.}$$

$$\lim_{t\to\infty}\frac{f(2^p t)}{f(2^p t)} \le \lim_{t\to\infty}\frac{f(mt)}{f(2^p t)} \le \lim_{t\to\infty}\frac{f(2^{p+1}t)}{f(2^p t)}$$

$$\implies \lim_{t\to\infty}\frac{f(mt)}{f(2^p t)} = 1$$

Now

$$\frac{f(mt)}{f(t)} = \frac{f(mt)}{f(2^p t)}\cdot\frac{f(2^p t)}{f(2^{p-1}t)}\cdots\frac{f(2t)}{f(t)}.$$

Each of the numbers in the limit is 1 and, therefore, $\lim_{n\to\infty}\frac{f(mt)}{f(t)} = 1$. Similarly argue for $0 < m < 1$.

32. First, notice that

$$0 \le |r - \sqrt{m^2+2n^2}| = \frac{|r^2-m^2-2n^2|}{r+\sqrt{m^2+2n^2}} \le \frac{|r^2-m^2-2n^2|}{r}.$$

Thus, to understand how close r can be to $\sqrt{m^2+2n^2}$, it suffices to estimate the final expression.

Choose the largest integer $m \ge 0$ such that $r^2 - m^2 \ge 0$. Then $m^2 \le r^2 < (m+1)^2$, which gives $m \le r$ and

$$r^2 - m^2 < 2m+1.$$

Next, we carry out the same idea one level deeper. Select the largest integer $n \ge 0$ satisfying $r^2 - m^2 - 2n^2 \ge 0$. Then

$$2n^2 \le r^2 - m^2 < 2(n+1)^2,$$

which shows that $n \leq \sqrt{(r^2 - m^2)/2}$. Now we can bound the difference:

$$\begin{aligned}
|r^2 - m^2 - 2n^2| &= r^2 - m^2 - 2n^2 \\
&< 2(2n+1) \\
&\leq 2 + 4\sqrt{\frac{r^2 - m^2}{2}} \\
&\leq 2 + \sqrt{2m+1} \\
&\leq 2 + \sqrt{2r+1}\,.
\end{aligned}$$

Therefore,

$$G(r) \leq \frac{2 + \sqrt{2r+1}}{r}.$$

Hence,

$$\lim_{r\to\infty} G(r) = 0.$$

Continuity and Differentiability

4. Claim: $P(x)$ must be a three-degree polynomial to satisfy the given condition.

$$c|n^3| \leq |P(n)| \leq d|n^3|, \text{ for all natural numbers } n$$

$$\text{Or, } c \leq \left|\frac{P(n)}{n^3}\right| \leq d, \text{ for all natural numbers.}$$

If P is a polynomial of degree 2 or fewer or degree 4 or more, then the inequality

$$c \leq \lim_{n\to\infty}\left|\frac{P(n)}{n^3}\right| \leq d$$

is violated. Therefore, $P(x)$ must be a polynomial of degree three and therefore must have at least one real root.

5. The given polynomial is

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n$$

$$\text{Let } M = \max\{1, 2n|a_{n-1}|, \ldots, 2n|a_0|\}$$

Then for all x with $|x| \geq M$, we have

$$\frac{1}{2} \leq 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}$$

$$\implies |f(x)| = \left|x^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right)\right| \geq \frac{x^n}{2}$$

If $c > M$ satisfies $|c^n| \geq 2f(0)$, then $|f(x)| \geq f(0)$ for $|x| \geq c$. So the minimum value of $|f(x)|$ on the interval $[-c, c]$ is the minimum value on $\mathbb{R}$.

6. The function is discontinuous at rationals. Indeed, let **a** = $\frac{p}{q}$ be rational number, such that $gcd(p,q) = 1$. So we have $f(a) = \frac{1}{q}$, while let $\{b_n\}$ be a sequence of irrationals that converges to a, then the limiting value of the function through this sequence is 0 and therefore the function is discontinuous at a. The function is continuous at irrational points. Say **c** is an irrational point. So we have $f(c) = 0$. Any sequence of irrational converging to c will have its limit as 0. And any sequence of rationals converging to c will have its limits as 0.

8. (d)
Say $x_1 \neq x_2 \implies |f(x_1) - f(x_2)| \geq \log(1 + |x_1 - x_2|) > 0 \implies f(x_1) \neq f(x_2)$. Therefore, f is one-one. To prove that f is onto we will prove that f is not bounded from above or below, the fact that f is continuous will compel f to take all real values. Say that f is bounded from above by M and say $f(0) = a$. Since f is continuous and one-one, so f must be strictly increasing or strictly

decreasing! WLOG, let f be strictly increasing. So, for $x > 0$, $f(x) > f(0) = a > 0 \implies f(x) > 0$ for all $x > 0$.

$$|f(x) - f(0)| \geq \log(1 + |x|) \implies f(x) - f(0) = |f(x) - f(0)| \geq \log(1 + |x|)$$
$$\implies f(x) - a \geq \log(1 + x) \implies f(x) \geq \log(1 + x) + a$$

So, if $x > e^{M-a} - 1$, then $f(x) > M$, which contradicts the assumption that f is bounded from above. Similarly, one can prove that f is not bounded from below. Since f is continuous, f must take all real values.

9. The equation can be rewritten as

$$tf(x) = \int_0^t (f(x+y) - f(y))\, dy$$

Differentiating this with respect to t

$$f(x) = f(x+t) - f(t)$$
$$f(x+t) = f(x) + f(t), \text{ for all } x \text{ and } t > 0.$$

Since f is continuous, so $f(x) = cx$ for some constant c.

10. We have $g'(x) = 2 \cdot \frac{f(2x)}{2x} - \frac{f(x)}{x} = \frac{f(2x)-f(x)}{x} = 0$. Hence g is constant.

12. For the sake of contradiction, let us assume that this is indeed true. And let $a < b$ be two points such that $f(a) = f(b)$. So in the interval $[a, b]$, f cannot take values which are larger than $f(a)$ and smaller than $f(a)$. So for all $x \in [a, b]$, let $f(x) < f(a)$(you can take the opposite). Now for $x < a$ and $x > b$ for the condition to hold $f(x) > f(a)$. This means that absolute minima attained on the interval $[a, b]$ is in fact the absolute minima on $\mathbb{R}$. So, f is bounded from below, this contradicts the fact that f takes every value twice.

14. Since f is differentiable, therefore f is continuous. Since $\lim_{n\to\infty} f(\frac{1}{n}) = 0 \implies f(0) = 0$. Since f is also continuous at 0.

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

Since we know that $f'(0)$ exists, because f is differentiable everywhere, it is differentiable at 0, so say $h = \frac{1}{n}$, where n is a natural number.

$$f'(0) = \lim_{n\to\infty} \frac{f(\frac{1}{n})}{\frac{1}{n}} = 0$$

Since $f(\frac{1}{n}) = 0$ for any natural number n. For other parts make a counterexample

$$f(x) = \begin{cases} 10x^2 \sin(\frac{1}{x}), x \neq 0 \\ 0, \text{when } x = 0. \end{cases}$$

21. For $x \neq y$, the inequality can be rephrased as

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq |x-y|$$

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \leq \lim_{y\to x}|x-y|$$

$$\implies |f'(x)| \leq 0$$

But $|f'(x)| < 0$ is not possible. $|f'(x)| = 0 = f'(x)$ or f is a constant. However, in doing so we have assumed f to be differentiable. Let us attempt to prove this without assuming that f is differentiable. If we can prove that for any $x \neq y$, $f(x) = f(y)$, then we are done.

WLOG say that $x < y$ and divide the interval $[x,y]$ into n equal parts. Now the given inequality

$$|f(x)-f(y)| \leq (x-y)^2$$

$$|f(x) - f(x+\frac{y-x}{n}) + f(x+\frac{1}{n}) - f(x+\frac{2(y-x)}{n}) + \cdots + f(x+\frac{(n-1)(y-x)}{n}) - f(y)|$$

$$\leq |f(x)-f(x+\frac{y-x}{n})| + \cdots + |f(x+\frac{(n-1)(y-x)}{n}) - f(y)|$$

$$\leq \sum_{i=0}^{n-1}\frac{(y-x)^2}{n^2} = \frac{(y-x)^2}{n} \text{ for all } n.$$

$$|f(x)-f(y)| \leq \lim_{n\to\infty}\frac{(y-x)^2}{n} = 0$$

$$\implies |f(x)-f(y)| = 0 \implies f(x) = f(y), \text{ for any } x \neq y.$$

$\implies f$ is constant.

22. Clearly, $f_n''(0) = -\sum_{i=0}^{n} i^2$.

$$\implies |f_n''(0)| = \frac{n(n+1)(2n+1)}{6}.$$

Since $\frac{n(n+1)(2n+1)}{6}$ is a strictly increasing function of n, so the least value of n is 18.

23.(a) Since $f(x^2) = (f(x))^2$. Put $x = 0$ to get $f(0) = (f(0))^2$. Since, $f(0) \neq 0 \implies f(0) = 1$. Since the domain of the function is set of nonnegative numbers, so using induction we get $f(x^{2^n}) = (f(x))^{2^n}$. So we have $\lim_{n\to\infty} f(x^{2^n}) = \lim_{n\to\infty}(f(x))^{2^n}$. Since $0 < x < 1 \implies \lim_{n\to\infty} x^{2^n} = 0 \implies \lim_{n\to\infty} f(x^{2^n}) = f(0) = 1$. For this to happen, the only nonnegative value of $f(x)$ is 1. So, the function is constant $f(x) = 1\ \forall x$.

(b) $f(x^2) = (f(x))^2 \implies f(x) = f(\sqrt{x})^2 \implies f(\sqrt{x}) = \sqrt{f(x)}$. Proceeding in the same way, we get $\sqrt[\frac{1}{2^n}]{f(x)} = f(x^{\frac{1}{2^n}})$. Suppose for some number $a, f(a) = 0$ and for some number $b, f(b) > 0$, then $f(a^{\frac{1}{2^n}})$ and $f(b^{\frac{1}{2^n}})$ both converge to $f(1)$ and both give different values of $f(1)$ which is a contradiction.
(c) Notice that $f(x) = x^m$ satisfies the function equation. Choose a suitable value of m to meet the required condition.

Mean Value Theorems

1. Consider the function $h(x) = f(x) - 3g(x)$. Obviously, the given function $h(x)$ is continuous in $[-1,2]$ and differentiable in $(-1,2)$. And further $h(-1) = h(0) = h(2) = 0$. Using Rolle's theorem, $h'(x)$ will be zero at least one in $(-1,0)$ and $(0,2)$. To prove that $h'(x)$ will vanish exactly once in those intervals, let us for instance assume that $h'(x)$ vanishes at least twice in the interval $(-1,0)$, but that will imply $h''(x)$ will vanish at least twice in the interval $(-1,0)$ which is a contradiction.

2. Define $g(x) = f(x) - x^2$. Obviously $g(x)$ is differentiable and therefore continuous. Clearly, $g(1) = g(2) = g(3) = 0$ and hence using Rolle's theorem $g'(x)$ must be zero at least once in $(1,2)$ and further $(2,3)$. And similarly, applying Rolle's theorem to $g'(x)$ we get $g''(x) = 0$, we get $f(x) = 2$ for at least one x in $(1,3)$.

6. Obviously if $c = 0$, all the roots have a multiplicity equal to 1 and the statement is true. However, let us examine the case when $c \neq 0$. Consider

$$f(x) = x(x+1)(x+2)\cdots(x+2009) - c$$
$$\implies f(-2009) = f(-2008) = \cdots = f(-1) = f(0) = -c.$$

Since f is a polynomial of degree 2010, so f' is a polynomial of degree 2009 and therefore can have at most 2009 zeros. But using Rolle's Theorem, it will have exactly 2009 zeros. Let them be $\alpha_1 < \alpha_2 < \cdots < \alpha_{2009}$. Using Rolle's theorem on f', we get that f'' will have exactly 2008 roots $\beta_1, \beta_2, \ldots, \beta_{2008}$ such that $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \beta_{2008} < \alpha_{2009}$. If $P(\gamma) = P'(\gamma) = 0$, then γ must be one of the α_i which is not equal to β_i and therefore $P''(\gamma) \neq 0$ and hence its multiplicity cannot be greater than or equal to 3.

8. Say that all the roots of the equation $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots a_{n-1}x + a_n = 0$, are real. So using Rolle's theorem the $(n-2)$th derivative of $f(x)$ that is $f^{(n-2)}(x) = 0$ must have at least two real roots. Since it is a polynomial of degree two, so it cannot have more than two roots. So $f^{(n-2)}(x) = 0$ must have exactly two real roots.

$$f^{(n-2)}(x) = \frac{n!}{2}x^2 + (n-1)!a_1x + (n-2)!a_2$$

This will have two real roots iff its discriminant is nonnegative.

$$(n-1)!^2a_1^2 - 2n!(n-2)!a_2 \geq 0$$
$$\implies (n-1)a_1^2 \geq 2na_2$$

But we have been given that $a_1^2 < a_2 \implies a_2 > 0$.

$$(n-1)a_1^2 \geq 2na_2 \implies a_1^2 \geq \frac{2n}{n-1}a_2 \geq 2a_2.$$

Which is a contradiction.

OR

Say that the roots of the equation $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_{n-1} x + a_n = 0$ are $\alpha_i, i = 1, 2, \ldots, n$ which are all real.

$$\sum_{i=1}^{n} \alpha_i = -a_1, \text{ and, } \sum_{1 \le i < j \le n} \alpha_i \alpha_j = a_2.$$

$$\text{Since } \alpha_i s \text{ are all real } \implies \sum_{i=1}^{n} \alpha_i^2 \ge 0$$

$$\sum_{i=1}^{n} \alpha_i^2 = \left(\sum_{i=1}^{n} \alpha_i \right)^2 - 2 \sum_{1 \le i < j \le n} \alpha_i \alpha_j = a_1^2 - 2a_2 < 0 \text{ (as } a_1^2 < a_2).$$

Which is a contradiction.

10. For the function $Q_n(x) = 1 - x^n - (1-x)^n$, obviously $Q_n(0) = Q_n(1) = 0$. Say that $Q_n(x) = 0$ has three zeros, then $Q_n'(x) = 0$ must have at least two.

$$Q_n'(x) = -nx^{n-1} + n(1-x)^{n-1} = 0.$$

If this equation has two roots (and notice that $x = 0$ is not a root), then $\left(\frac{x}{1-x}\right)^{n-1} = 1$ for at least two real values of x. This is only possible if $\frac{x}{1-x} = \pm 1$. But $\frac{x}{1-x} \neq -1$ for any real value of x and therefore we have reached a contradiction.

11. Obviously, the equation

$$4^x + 6^{x^2} = 5^x + 5^{x^2}$$

has $x = 0, 1$ satisfying it. Next, we claim that there is no other value of x for which this is true. Consider the function,

$$f(t) = t^{x^2} + (10-t)^x$$

Clearly, $f(5) = f(6)$. The function satisfies all the conditions of Rolle's Theorem, so we have $f'(c) = 0$.

$$\implies xc^{x^2-1} - x(10-c)^{x-1} = 0, \text{ or } xc^{x^2-1} = (10-c)^{x-1}.$$

Since the bases of all the exponentials are positive, x must be positive. If $x > 1$, then $xc^{x^2-1} > c^{x^2-1} > c^{x-1} > (10-c)^{x-1}$, which is impossible since we have $xc^{x^2-1} = (10-c)^{x-1}$.

If $0 < x < 1$, then $xc^{x^2-1} < xc^{x-1}$. Now $x < \left(\frac{10-c}{c}\right)^{x-1}$. Because on the left hand side we have an increasing function and on right hand side we have an exponentially decreasing function whose base is less than one, and the two intersect at $x = 1$. Therefore the inequality $xc^{x-1} < (10-c)^{x-1}$ follows. So we conclude that $xc^{x-1} \neq (10-c)^{x-1}$. So, a third solution does not exists and therefore the only given solutions are $x = 0, 1$.

12. Consider the function $g(x) = f(x)\cos x$. Since g is continuous in $[\frac{-\pi}{2}, \frac{\pi}{2}]$ and differentiable in $(\frac{-\pi}{2}, \frac{\pi}{2})$ and $f(\frac{-\pi}{2}) = f(0) = f(\frac{\pi}{2}) = 0$. So, using Rolle's Theorem $f'(c_1) = f'(c_2) = 0$. Now define a function $h(x) = \frac{g'(x)}{\cos^2 x}$ and use Rolle's theorem on $h(x)$ in the interval $[c_1, c_2]$ to get the desired result.

13. Since f is differentiable in $(0,2) \cup (4,6)$. So, invoking Lagrange's mean value theorem we get:

$$\frac{f(1.5) - f(0.5)}{1.5 - 0.5} = f'(c) = 1$$

$$\text{And similarly, } \frac{f(5.5) - f(4.5)}{5.5 - 4.5} = f'(d) = 1$$

$$f(1.5) - f(0.5) = f(5.5) - f(4.5).$$

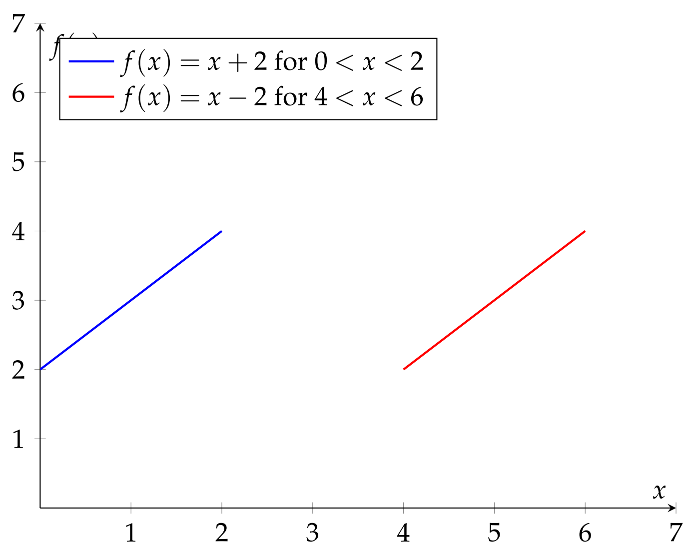
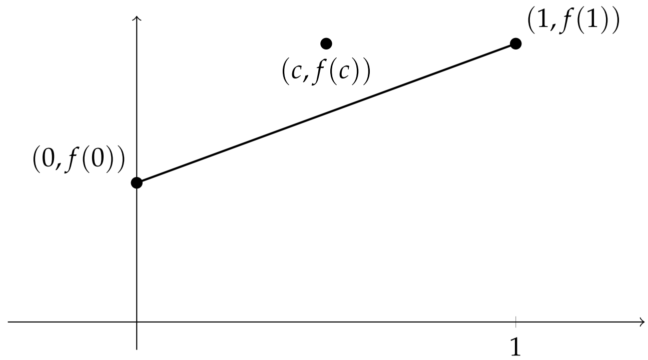


Figure: The blue line shows $f(x) = x + 2$ on the interval $(0,2)$, and the red line shows $f(x) = x - 2$ on the interval $(4,6)$.

15. So, either the function defined on $[0,1]$ is linear and we get $f(x) = ax + b$ or there is some $c \in (0,1)$ for which $f(c)$ does not lie on the line joining the points $(0, f(0))$ and $(1, f(1))$. So, $f(c)$ must lie above the line segment joining the points $(0, f(0))$ and $(1, f(1))$ or below the line segment. So, WLOG let $(c, f(c))$



be above the line segment. Notice that the quantity $\frac{f(1)-f(0)}{1-0}$ represents the slope of the line joining the points $(0, f(0))$ and $(1, f(1))$. Since $(c, f(c))$ lies above the line segment, therefore, the magnitude of slope of the line segment joining the points $(0, f(0))$ and $(c, f(c))$ is greater than the magnitude of slope of the line segment joining the points $(0, f(0))$ and $(1, f(1))$.

$$\implies \left|\frac{f(1)-f(0)}{1-0}\right| < \left|\frac{f(c)-f(0)}{c-0}\right|$$

Invoking Lagrange's mean value theorem on f

$$\frac{f(c)-f(0)}{c-0} = f'(t) \implies \left|\frac{f(c)-f(0)}{c-0}\right| = |f'(t)| \text{ for some } \ t \in (0,c).$$

$$\implies \left|\frac{f(1)-f(0)}{1-0}\right| < |f'(t)| \text{ for some } t.$$

16. Say for the sake of assumption that between any two successive zeros of g, there is no zero of f. Say that the two successive zeros of g are α_1 and α_2. Since f is differentiable, therefore it is continuous and since f is not zero, therefore it should be positive throughout the interval or negative throughout the interval. So, consider the function $h(x) = \frac{g(x)}{f(x)}$. It is continuous in the interval $[\alpha_1, \alpha_2]$. It is differentiable in the interval (α_1, α_2) and $h(\alpha_1) = h(\alpha_2) = 0$. Therefore, invoking Rolle's theorem we have: there is some $c \in (\alpha_1, \alpha_2)$ such that $h'(c) = 0$ or $\frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2} = 0 \implies g(c)f'(c) - f(c)g'(c) = 0$. This is a contradiction, and therefore our earlier assumption was incorrect, and f must be zero at least once between two consecutive zeros of g.

18. If $f(a) = f(b)$, then f satisfies all the conditions for Rolle's theorem, and therefore there exists some $\xi \in (a, b)$ such that $f'(\xi) = 0$. If $f(a) \neq f(b)$, then WLOG, let $f(b) > f(a)$. Since we have $\frac{f(b)-f(c)}{f(c)-f(a)} < 0$, so $f(c) < f(a) < f(b)$ or equivalently $(f(a) < f(b) < f(c))$. Consider any one of the cases, say

that $f(c) < f(a) < f(b)$. Since $a < c < b$ and $f(c) < f(a) < f(b)$, using the intermediate value theorem, there exist some $\chi \in (c,b) : f(\chi) = f(a)$. So, using Rolle's theorem on f in the interval $[a,\chi]$, we get that there exists some $\xi \in (a,\chi) : f'(\xi) = 0$. Or equivalently, we could have argued that since $f(c) < f(a) < f(b)$ and $a < c < b$, therefore, f attains its absolute minima at an interior point of the interval$[a,b]$. Let this point be $\xi \implies f'(\xi) = 0$. Or, condition $\frac{f(b)-f(c)}{f(c)-f(a)} < 0$ can be rephrased as $\frac{f(b)-f(c)}{b-c} \cdot \frac{c-a}{f(c)-f(a)} < 0$. Using Lagrange's mean value theorem, we have $\frac{f(c)-f(a)}{c-a} = f'(c_1)$ for some $c_1 \in (a,c)$ and $\frac{f(b)-f(c)}{b-c} = f'(c_2)$ for some $c_2 \in (c,b)$. So, now we have $\frac{f'(c_2)}{f'(c_1)} < 0$. This implies $f'(c_1)$ and $f'(c_2)$ are of opposite signs, so there exist some $\xi \in (c_1,c_2)$ such that $f'(\xi) = 0$(Darboux's theorem).

19. The given equation

$$x2^y + y2^{-x} = x + y,$$

can be rephrased as:

$$\begin{aligned}
x2^{x+y} + y &= 2^x(x+y) \\
x(2^{x+y} - 2^x) &= y(2^x - 1) \\
\frac{2^{x+y} - 2^x}{y} &= \frac{2^x - 1}{x} \\
\implies \frac{2^{x+y} - 2^x}{(x+y) - x} &= \frac{2^x - 2^0}{x - 0} \qquad \text{(A)}
\end{aligned}$$

Consider the function $f(t) = 2^t \implies f'(t) = 2^t \log 2$. Since the function f meets all the criteria for Lagrange's mean value theorem, we have:

$$\implies \frac{2^x - 2^0}{x - 0} = 2^{c_1} \log 2, \text{ for some } \ c_1 \in (0,x) \qquad \text{(B)}$$

$$\text{Similarly,} \frac{2^{x+y} - 2^x}{(x+y) - x} = 2^{c_2} \log 2 \text{ for some } \ c_2 \in (x, x+y) \qquad \text{(C)}$$

However, the function $2^u \log 2$ is strictly increasing, and therefore (B) and (C) will never be equal. Therefore, (A) does not hold.

22. Since

$$\begin{aligned}
f'(x) - f'(y) &\leq 3|x-y| \\
\implies |f'(x) - f'(y)| &\leq 3|x-y|
\end{aligned}$$

For $x = y$ the inequality is trivially true. So, WLOG let $y < x$

$$\int_y^x f'(x)dx - \int_y^x f'(y)\, dx \leq 3\int_y^x |x-y|\, dx$$

$$\implies f(x) - f(y) - f'(y)(x-y) \leq \frac{3}{2}(x-y)^2$$

$$\implies |f(x) - f(y) - f'(y)(x-y)| \leq \frac{3}{2}(x-y)^2$$

$$\text{Since, } |f'(x) - f'(y)| \leq 3|x-y|$$

$$\implies \frac{|f'(x) - f'(y)|}{|x-y|} \leq 3, \text{ for } y \neq x.$$

$$\implies \lim_{y \to x} \frac{|f'(x) - f'(y)|}{|x-y|} \leq 3$$

$$\implies |f''(x)| \leq 3.$$

So, the maximum value of $f''(x)$ is 3.

Application of Derivatives

2. Write
$$f(x) = \ln(1 + x^2).$$
We show that $y = ax + b$ intersects $y = f(x)$ in exactly one point if and only if (a,b) lies in one of the following groups:

- $a = b = 0$,
- $|a| \geq 1$, arbitrary b,
- $0 < |a| < 1$, and
$$b < \ln(1 - r_-^2) - |a|r_- \quad \text{or} \quad b > \ln(1 - r_+^2) - |a|r_+,$$
where
$$r_\pm = \frac{1 \pm \sqrt{1 - a^2}}{a}.$$

Since the graph of $y = f(x)$ is symmetric under reflection in the y-axis, it suffices to consider the case $a \geq 0$: $y = ax + b$ and $y = -ax + b$ intersect $y = f(x)$ the same number of times.

For $a = 0$, by the symmetry of $y = f(x)$ and the fact that $f(x) > 0$ for all $x \neq 0$, the only line $y = b$ that intersects $y = f(x)$ exactly once is the line $y = 0$.

We next observe that on $[0, \infty)$,
$$f'(x) = \frac{2x}{1 + x^2}$$
increases on $[0,1]$ from $f'(0) = 0$ to a maximum at $f'(1) = 1$, and then decreases on $[1, \infty)$ with $\lim_{x \to \infty} f'(x) = 0$. In particular, $f'(x) \leq 1$ for all x (including $x < 0$ since then $f'(x) < 0$) and $f'(x)$ achieves each value in $(0,1)$ exactly twice on $[0, \infty)$.

For $a \geq 1$, we claim that any line $y = ax + b$ intersects $y = f(x)$ exactly once. They must intersect at least once by the intermediate value theorem: for $x \ll 0$, $ax + b < 0 < f(x)$, while for $x \gg 0$, $ax + b > f(x)$ since
$$\lim_{x \to \infty} \frac{\ln(1 + x^2)}{x} = 0.$$
On the other hand, they cannot intersect more than once: for $a > 1$, this follows from the mean value theorem, since $f'(x) < a$ for all x. For $a = 1$, suppose that they intersect at two points (x_0, y_0) and (x_1, y_1). Then
$$1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} f'(x)\, dx < 1$$
since $f'(x)$ is continuous and $f'(x) \leq 1$ with equality only at one point.

Finally, consider $0 < a < 1$. The equation $f'(x) = a$ has exactly two solutions, at $x = r_-$ and $x = r_+$ for $r_\pm$ as defined above. If we define

$$g(x) = f(x) - ax,$$

then $g'(r_\pm) = 0$; g' is strictly decreasing on $(-\infty, r_-)$, strictly increasing on (r_-, r_+), and strictly decreasing on (r_+, ∞); and

$$\lim_{x\to-\infty} g(x) = \infty, \quad \lim_{x\to\infty} g(x) = -\infty.$$

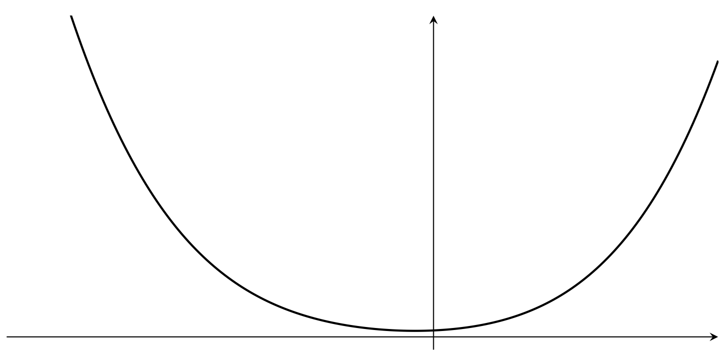
It follows that $g(x) = b$ has exactly one solution for $b < g(r_-)$ or $b > g(r_+)$, exactly three solutions for $g(r_-) < b < g(r_+)$, and exactly two solutions for $b = g(r_\pm)$.

That is, $y = ax + b$ intersects $y = f(x)$ in exactly one point if and only if

$$b < g(r_-) \quad \text{or} \quad b > g(r_+).$$

3. For reasons quite obvious, no vertical line can intersect the curve $y = x^4 + 9x^3 + cx^2 + 9x + 4$ twice, let alone four times. So we are interested in lines having a real slope. Since the curve is a degree four polynomial so it will have at least one point of absolute minima. Since the leading coefficient of the polynomial is positive so, the polynomial will either have exactly one minima and no maxima or it will have two minima and one maxima. So we can have exactly one of the two cases:

- When the curve has exactly one point of minima that is $f'(x) = 0$ will have exactly one root and $f''(x) = 0$ will have no root, or the curve will have no points of inflections.

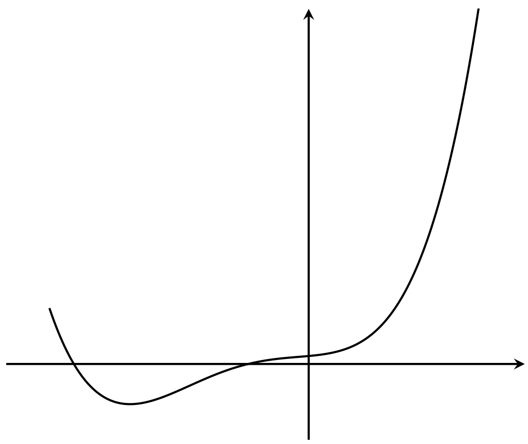
  Note that if the we have such a curve then any straight line will intersect the curve in a maximum of two points.

- When the curve has two points of minima and one point of maxima that is $f'(x) = 0$ will have exactly three real roots, and the curve will have real points of inflection.

  $f'(x) = 4x^3 + 27x^2 + 2cx + 9 \implies f''(x) = 12x^2 + 54x + 2c = 2(6x^2 + 27x + c)$

And for the polynomial to have real points of inflection the equation $f''(x) = 0$ must have real and distinct roots. Or the discriminant must be positive.

$$\implies (27)^2 - 4 \cdot 6 \cdot c > 0 \implies c < \frac{243}{8}.$$



As it can be seen that a line can be found out to intersect the curve in four distinct points. So if $c < \frac{243}{8}$ there exist lines which intersect the curve in four distinct points.

4. If $f(x) = x^3 + x^2 + cx + d$, then $f'(x) = 3x^2 + 2x + c$. If $c > \frac{1}{3}$, then notice that the discriminant of the quadratic $(2)^2 - 4 \cdot 3 \cdot c = 12(\frac{1}{3} - c) < 0$. Therefore, refer to chapter-1, $f'(x) > 0$ for all $x \in \mathbb{R}$. So, $f(x)$ intersects x-axis only once.

5. Since f is increasing and g is decreasing, so $f(g(x))$ is decreasing. Therefore, for any $x > 0$ we have $f(g(x)) \leq f(g(0)) = 0$. but $f(g(x))$ must take values from the co-domain of $f \implies f(g(x)) = 0$ for all x. Therefore, we have $h(x) = 0$ for all x, $h(x) - h(1) = 0$ for all x.

6. A return to the familiar trick of integrating factor! Since $f'(x) > 2f(x)$

$$f'(x) - 2f(x) > 0 \implies e^{-2x}f'(x) - 2e^{-2x}f(x) > 0$$

$$\frac{d}{dx}(e^{-2x}f(x)) > 0$$

$\implies g(x) = e^{-2x}f(x)$ is a strictly increasing function. So for $x > 0$, $g(x) > g(0) = 1$.

$$\implies e^{-2x}f(x) > 1 \implies f(x) > e^{2x} \text{ for } x \in (0, \infty).$$

Since $f'(x) > 2f(x) > 2e^{2x} > 0$, therefore f is increasing!

8. Since $f''(x) \geq f(x) \implies f''(x) - f(x) \geq 0$

$$\implies (f''(x) - f'(x)) + (f'(x) - f(x)) \geq 0$$

Say $g(x) = (f'(x) - f(x)) \implies g'(x) + g(x) \geq 0$.

$$e^x g'(x) + e^x g(x) \geq 0 \implies (e^x g(x))' \geq 0.$$

$e^x g(x)$ is an increasing function. $\implies e^x g(x) \geq e^0 g(0) > 0$ for all $x > 0$. $\implies$ $g(x) > 0$ for all $x > 0$.

$$f'(x) - f(x) > 0 \implies e^{-x} f'(x) - e^{-x} f(x) > 0 \implies (e^{-x} f(x))' > 0.$$

The function $e^{-x} f(x)$ is strictly increasing. $e^{-x} f(x) > e^{-0} f(0) = 0$ for all $x > 0$. So we have $f(x) > 0$ for all $x > 0$.

11. Since $f''(x) + 2f'(x) + f(x) \geq 0 \implies e^x f''(x) + 2e^x f'(x) + e^x f(x) \geq 0$

$$\frac{d^2}{dx^2}(e^x f(x)) \geq 0.$$

So, the curve $g(x) = e^x f(x)$ is convex in the interval $[0,1]$ and $f(0) = f(1) = 0$. So, we have $f(x) \leq 0$ for all $x \in [0,1]$.

12. The given equation is $y'' - 2y' + y = 2e^x$ which can be rephrased as

$$e^{-x} y'' - 2e^{-x} y' + e^{-x} y = 2$$

$$\implies \frac{d^2}{dx^2}(e^{-x} y) = 2.$$

$$e^{-x} y = x^2 + ax + b, \text{ where } a, b \text{ are constants.}$$

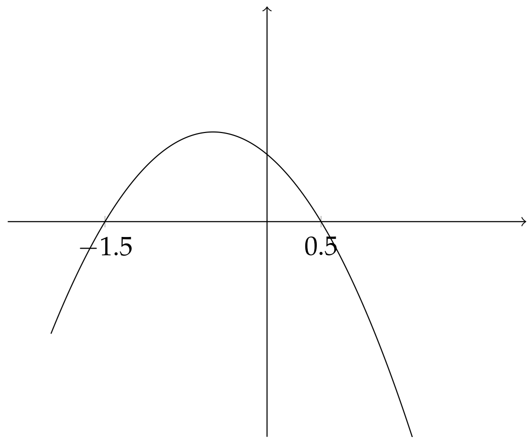
$$\implies y = e^x (x^2 + ax + b) \implies f(x) = e^x (x^2 + ax + b).$$

$$\text{Since} f(x) > 0 \text{ for all } x \implies a^2 - 4b < 0.$$

Now $f'(x) = e^x (x^2 + (a+2)x + a + b)$, which is positive for all x when $(a+2)^2 - 4(a+b) < 0 \implies a^2 + 4 - 4b < 0$. Note if $a^2 + 4 - 4b < 0$, then $a^2 - 4b < 0$. Therefore, if $f'(x) > 0 \implies f(x) > 0$.

13. Tweaking Minkowski's inequality $(1+x)^r \leq 1 + x^r$ for $0 \leq x \leq 1$ and $0 \leq r \leq 1$. So $\frac{(1+x)^{0.3}}{1+x^{0.3}} \leq \frac{1+x^{0.3}}{1+x^{0.3}} = 1$. So, the maximum value of $\frac{(1+x)^{0.3}}{1+x^{0.3}}$ is 1 and occurs at $x = 0$.

14.



Clearly the integral $\int_a^b (\frac{3}{4} - x - x^2)\, dx$ attains the largest value when $a = -1.5$ and $b = 0.5$.

15. The shortest distance is $\sqrt{5^2 + 2^2} = \sqrt{29}$.

16. Obviously g is a polynomial of degree 2 and therefore cannot be strictly increasing or decreasing. Since the leading coefficient of g is positive, therefore g must attain an absolute minimum. Let this be attained at $x = a \implies g'(a) = 0$.

$$\text{But } g'(a) = f''(a) - f'''(a) + f''''(a) = 0. \text{ However, } f''''(a) = 0$$

$$\implies f''(a) - f'''(a) = 0.$$

$$g(a) = f'(a) - f''(a) + f'''(a) = f'(a) > 0. \text{ (Since } f \text{ is strictly increasing!)}$$

But a was the point of absolute minima, therefore $g(x) > 0$ for all x.

18. Since $f'(x) = \frac{1}{x^2 + (f(x))^2}$. So, $f'(x) > 0$ and f is strictly increasing. Therefore, $f(x) \geq f(1) = 1$ for $x > 1$. Therefore, for any $x > 1$

$$f'(x) = \frac{1}{x^2 + (f(x))^2} \leq \frac{1}{1 + x^2}$$

$$\implies \int_1^x f'(x)dx \leq \int_1^x \frac{1}{1 + x^2}dx$$

$$\implies f(x) \leq 1 + \tan^{-1} x - \frac{\pi}{4} \text{ for all x} > 0.$$

$$\implies f(x) \leq \lim_{x\to\infty} (1 + \tan^{-1} x - \frac{\pi}{4}) = 1 + \frac{\pi}{4}.$$

19. Since $f(x) > 0$, we therefore have:

$$\frac{\sqrt{f'(t)}}{\sqrt{f(t)}} \geq 1.$$

$$\int_1^x \frac{\sqrt{f'(t)}}{\sqrt{f(t)}} dt \leq \int_1^x dt = (x-1)$$

$$\implies \sqrt{f(x)} \geq \sqrt{f(1)} + \frac{1}{2}(x-1).$$

20. Since $f'(x)$ is differentiable, it must therefore be continuous. Since it is negative somewhere and positive somewhere, therefore by intermediate value theorem it must be 0 somewhere. Let this point be p. Say $f'(p) = 0$. Since $f''(x) > 0$ for all x, therefore $f'(x)$ is strictly increasing and $f'(x) > 0$ for all $x > p$, which is where the function is strictly increasing!

21. If f is monotonous in the interval, then $|f(x)| \leq 1$ for all $x \in [-1,1]$. And we are done. Or say that f is not monotonous and there is some $x_0 \in (a,b)$ such that $f'(x_0) = 0$. If we can show that $f(x_0) \leq 1$, then we are done. Also, notice that $|c| \leq 1$

$$|a+b+c| \leq 1, \text{ and } |a-b+c| \leq 1 \implies |2b| = |(a+b+c) - (a-b+c)| \leq$$

$$|a-b+c| + |a+b+c| \leq 2 \implies |b| \leq 1.$$

Moreover, $2ax_0 + b = 0$.

$$\implies f(x_0) = ax_0^2 + bx_0 + c = x_0(ax_0) + bx_0 + c = \frac{bx_0}{2} + c$$

$$\implies |f(x_0)| \leq \left|\frac{bx_0}{2}\right| + |c| \leq \frac{1}{2}|b||x_0| + |c| \leq \frac{1}{2} \cdot 1 \cdot 1 + 1 = \frac{3}{2}.$$

In fact we can prove a stronger result that $|\mathbf{f}(\mathbf{x})| \leq \frac{\mathbf{5}}{\mathbf{4}}$ for all $\mathbf{x} \in [-\mathbf{1},\mathbf{1}]$.

Define

$$p = f(-1), \quad q = f(0), \quad r = f(1).$$

By assumption,

$$|p| \leq 1, \quad |q| \leq 1, \quad |r| \leq 1.$$

We have

$$c = q, \quad b = \frac{r-p}{2}, \quad a = \frac{p+r-2q}{2}.$$

The function can be expressed using Lagrange interpolation:

$$f(x) = p \cdot \frac{x(x-1)}{2} + q \cdot (1-x^2) + r \cdot \frac{x(x+1)}{2}.$$

For fixed $x \in [-1,1]$, define

$$A(x) = \frac{x(x-1)}{2}, \quad B(x) = 1 - x^2, \quad C(x) = \frac{x(x+1)}{2}.$$

Thus,

$$f(x) = A(x)p + B(x)q + C(x)r.$$

Since $|p|,|q|,|r| \leq 1$ and $f(x)$ is linear in p,q,r,

$$|f(x)| \leq |A(x)| + |B(x)| + |C(x)| =: g(x).$$

We must show $g(x) \leq \frac{5}{4}$ for all $x \in [-1,1]$.

Case 1: $x \in [0,1]$.

We note:

$$x(x-1) \leq 0 \quad \Rightarrow \quad |A(x)| = \frac{x(1-x)}{2},$$
$$1 - x^2 \geq 0 \quad \Rightarrow \quad |B(x)| = 1 - x^2,$$
$$x(x+1) \geq 0 \quad \Rightarrow \quad |C(x)| = \frac{x(x+1)}{2}.$$

Thus:

$$g(x) = \frac{x(1-x)}{2} + (1 - x^2) + \frac{x(x+1)}{2} = -x^2 + x + 1.$$

This quadratic opens downwards, with vertex at $x = \frac{1}{2}$:

$$g\left(\frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{5}{4}.$$

At $x = 0$ and $x = 1$, we get $g = 1$. Hence $g(x) \leq \frac{5}{4}$.

Case 2: $x \in [-1,0]$.

We have:

$$x(x-1) \geq 0 \quad \Rightarrow \quad |A(x)| = \frac{x(x-1)}{2},$$
$$1 - x^2 \geq 0 \quad \Rightarrow \quad |B(x)| = 1 - x^2,$$
$$x(x+1) \leq 0 \quad \Rightarrow \quad |C(x)| = -\frac{x(x+1)}{2}.$$

Thus:

$$g(x) = \frac{x(x-1)}{2} + (1 - x^2) - \frac{x(x+1)}{2} = -x^2 - x + 1.$$

Vertex at $x = -\frac{1}{2}$:

$$g\left(-\frac{1}{2}\right) = -\frac{1}{4} + \frac{1}{2} + 1 = \frac{5}{4}.$$

At $x = -1$ and $x = 0$, we get $g = 1$. Hence $g(x) \leq \frac{5}{4}$.

Conclusion:

For all $x \in [-1,1]$, $g(x) \leq \frac{5}{4}$, so

$$|f(x)| \leq \frac{5}{4}.$$

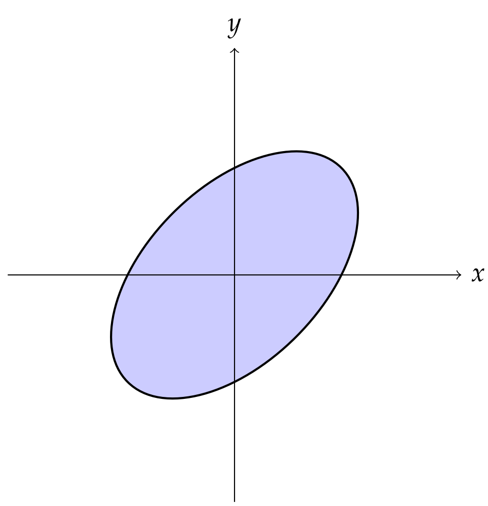
Equality occurs, for example, for $f(x) = -x^2 + x + 1$ at $x = \frac{1}{2}$ or $f(x) = -x^2 - x + 1$ at $x = -\frac{1}{2}$.

23. Say that $f(x)g(x)$ is indeed linear. So, we have

$$\begin{aligned} f(x)g(x) = ax + b &\implies f'(x)g(x) + f(x)g'(x) = a \\ &\implies f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) = 0 \\ &\implies f^2(x) + 2f'(x)g'(x) + g^2(x) = 0. \end{aligned}$$

But we have $f'(x), g'(x) \geq 0$, so $f(x) = g(x) = 0$.

24. Clearly, it represents all the points inside the region represented by the ellipse $x^2 + 2y^2 - 2xy = 1$ whose centre is the origin $(0,0)$.



Let there be a point on it $(r\cos\theta, r\sin\theta)$. Clearly, the length of the longest line segment is $2|r|$. The point lies on the ellipse; therefore,

$$r^2(\cos^2\theta + 2\sin^2\theta - 2\sin\theta\cos\theta) = 1.$$

$$r^2 = \frac{2}{3-\cos 2\theta - 2\sin 2\theta} \leq \frac{2}{3-\sqrt{5}} \implies r \leq \sqrt{\frac{2}{3-\sqrt{5}}}.$$

Therefore, the maximum length of the chord is $2\sqrt{\frac{2}{3-\sqrt{5}}}$.

26. Since the function is continuous in the closed interval $[a,b]$, therefore, it must attain an absolute maximum and minimum. If both maximum and the minimum are attained at the end points, then the function is a constant and $f(x) = 0$ for all x. If one of the point, say, maxima is attained at an internal point, say x_1, then $f'(x_1) = 0 \implies f(x_1) = f''(x_1)$. Since x_1 is a point of maxima, therefore $f''(x_1) \leq 0 \implies f(x_1) \leq 0$. By mimicking the same argument, if x_2 is a point of absolute minimum, then $f(x_2) \geq 0$. This can only happen if $f(x_1) = f(x_2) = 0$ and the function is constant.

27. If $f(x) = ax + \frac{1}{x+1} \implies f'(x) = a - \frac{1}{(x+1)^2}$ for all $x \in [0,1] \implies \frac{1}{4} \leq \frac{1}{(x+1)^2} \leq 1$.

- If $a \geq 1$, then $f'(x) \geq 0$ and the function is strictly increasing. So, we have $L - S = f(1) - f(0) = a + \frac{1}{2} - 1 = a - \frac{1}{2} > \frac{1}{12}$.
- If $a \leq \frac{1}{4}$, then $f'(x) \leq 0$ and the function is strictly decreasing. So, we have $L - S = f(0) - f(1) = 1 - (a + \frac{1}{2}) = \frac{1}{2} - a \geq \frac{1}{4} > \frac{1}{12}$.
- If $\frac{1}{4} < a < \frac{1}{2}$, then $f'(x) = 0$ for $x = \frac{1}{\sqrt{a}} - 1$ which is a point of minima. And $\frac{1}{\sqrt{a}} - 1 = 2\sqrt{a} - a$. The function is decreasing in $(0, \frac{1}{\sqrt{a}} - 1)$ and increasing in $(\frac{1}{\sqrt{a}} - 1, 1)$. So the largest value which $f(x)$ assumes on the interval is either $f(1) = a + \frac{1}{2}$ or $f(0) = 1$. Obviously $f(0) > f(1) \implies L - S = 1 - (2\sqrt{a} - a) = (1 - \sqrt{a})^2 > (1 - \frac{1}{\sqrt{2}})^2 = \frac{3-2\sqrt{2}}{2} = \frac{18-12\sqrt{2}}{12} > \frac{1}{12}$.
- If $a = \frac{1}{2}$, then $f(0) = f(1) = 1 = L$ and $S = \sqrt{2} - \frac{1}{2}$. So we have $L - S = \frac{3}{2} - \sqrt{2} > \frac{1}{12}$.
- Finally, if $\frac{1}{2} < a < 1$, then using the analysis of part III, we have $L = a + \frac{1}{2}$ and $S = 2\sqrt{a} - a$. So we have $L - S = 2a - 2\sqrt{a} + \frac{1}{2} = 2(\sqrt{a} - \frac{1}{2})^2 > \frac{1}{12}$.

And we are done.

28. Since $f(x) = x^n + x^{n-1} + \cdots + x + 1$

$$\implies f'(x) = nx^{n-1} + (n-1)x^{n-2} + \cdots + 2x + 1$$

Notice for any $x \geq 0$, $f'(x) > 0$. So f cannot have any positive roots. So critical points, if any, must be negative. For $x < 0$, the series is an AGP(though it is an AGP for $x \geq 0$ too) whose sum can be evaluated as:

$$f'(x) = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

Notice that if n is odd, then $nx^{n+1}-(n+1)x^n+1>0$ and therefore there are no critical points. However, if n is even, then for $g(x)=nx^{n+1}-(n+1)x^n+1$, we have $g(-1)<0$ and $g(0)>0$, so there is at least one root of $g(x)=0$ or $f'(x)=0$ in the interval $(-1,0)$. However, $g'(x)=n(n+1)x^n-n(n+1)x^{n-1}>0$ for $x\in(0,1)$. So, there is exactly one root of $f'(x)=0$ in the interval $(0,1)$. Notice that for $x<-1$, we have $f'(x)<0$. So, even in this case there is exactly one root of $f'(x)=0$ which lies between $(-1,0)$.

29. Let $z=x+iy$, where $i=\sqrt{-1}$.

$$|z^3-z+2|=|(x+iy)^3-(x+iy)+2|=|(x^3-3xy^2-x+2)+i(3x^2y-y^3-y)|$$
$$=\sqrt{(x^3-3xy^2-x+2)^2+(3x^2y-y^3-y)^2}$$

Since $|z|=1 \implies x^2+y^2=1$. So let $x=\cos\theta$ and $y=\sin\theta$.

$$\sqrt{(x^3-3xy^2-x+2)^2+(3x^2y-y^3-y)^2}=4(2+4\cos^2\theta-\cos^2\theta-4\cos\theta).$$

Since $-1\le\cos\theta\le 1$, therefore the maximum value is $\sqrt{13}$.

30. Apply the analysis of question 26.

31. Since $a\ge b\ge c \implies A\ge B\ge C$. Say that the circum-radius of the triangle is R. Since the sine function is concave on $[0,\pi]$, therefore we have

$$\frac{\sin B-\sin C}{B-C}\ge\frac{\sin A-\sin C}{A-C}$$
$$\implies (A-C)(\sin B-\sin C)\ge(B-C)(\sin A-\sin C)$$
$$\implies A\sin B-A\sin C-C\sin B\ge B\sin A-C\sin A-B\sin C$$

Rearranging the result and using sine rule we get the required inequality!

32. Consider the function $f(x)=x^2+\sqrt{x}$. Since $f''(x)=2-\frac{1}{4}x^{\frac{-3}{2}}$. So we have $f''(x)\ge 0$ for $x\ge\frac{1}{4}$. Therefore f is convex on the interval $[\frac{1}{4},\infty)$.

$$\implies \frac{f(x_1)+f(x_2)}{2}\ge f\left(\frac{x_1+x_2}{2}\right)$$

Substitute $x_1=a^2$ and $x_2=b^2$ to get the result.

33. Suppose for all i that

$$\Delta i=f\left(\tfrac{i}{n}\right)-f\left(\tfrac{i-1}{n}\right).$$

We are given that

$$f\left(\tfrac{i}{n}\right)+f\left(\tfrac{i+2}{n}\right)\ \ge\ 2f\left(\tfrac{i+1}{n}\right)+\tfrac{4}{n}.$$

This implies

$$f\left(\tfrac{i+2}{n}\right) - f\left(\tfrac{i+1}{n}\right) \;\geq\; f\left(\tfrac{i+1}{n}\right) - f\left(\tfrac{i}{n}\right) + \tfrac{4}{n}.$$

In other terms,

$$\Delta(i+1) \;\geq\; \Delta i + \tfrac{4}{n}.$$

If we use this inequality for n consecutive values of i, we obtain

$$\Delta(i+n) \;\geq\; \Delta i + 4.$$

Summing this for $i = 1, 2, \ldots, n$, we get

$$f(2) - f(1) \;\geq\; f(1) - f(0) + 4n.$$

Clearly, this inequality cannot hold true for all n, and thus we are done.

34. Apply Jensen's Inequality on $f(x) = \sqrt{x}$ and use the weights $\lambda_1 = \frac{1}{10}$, $\lambda_2 = \frac{2}{10}, \lambda_3 = \frac{3}{10}, \lambda_4 = \frac{4}{10}$ to get the result.

35. Consider the function $f(x) = \ln\left(\frac{\sin x}{x}\right)$ on the interval $(0, \pi)$. We have $f''(x) = \frac{-1}{\sin^2 x} + \frac{1}{x^2} < 0$. Therefore, f is concave in the interval. Take natural log and the inequality follows.

36. Since the triangle is convex, A can be expressed as a convex combination of its vertices:

$$A = \alpha P + \beta Q + \gamma R,$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$.

If $P = (x_P, y_P), Q = (x_Q, y_Q), R = (x_R, y_R)$, then

$$f(A) = a(\alpha x_P + \beta x_Q + \gamma x_R) + b(\alpha y_P + \beta y_Q + \gamma y_R) + c.$$

Rearranging gives

$$f(A) = \alpha(ax_P + by_P + c) + \beta(ax_Q + by_Q + c) + \gamma(ax_R + by_R + c).$$

Thus

$$f(A) = \alpha f(P) + \beta f(Q) + \gamma f(R).$$

Since $\alpha, \beta, \gamma \geq 0$ and sum to 1, $f(A)$ is a weighted average of $f(P), f(Q), f(R)$. Therefore,

$$f(A) \leq \max\{f(P), f(Q), f(R)\}.$$

This completes the proof.

38. Since $P(x) > 0$ for all $x \in \mathbb{R}$, therefore n is even and the leading coefficient is positive. Consider the polynomial

$$Q(x) = P(x) + P'(x) + \cdots + P^n(x)$$

Which is a polynomial of even degree with positive leading coefficient and therefore it must have a point of absolute minimum **a**, where $Q'(a) = 0$.

$$Q'(a) = P'(a) + P''(a) + \cdots + P^n(a) + P^{n+1}(a) = 0$$

$$Q(a) = P(a) + P'(a) + \cdots + P^n(a) = P(a) > 0$$

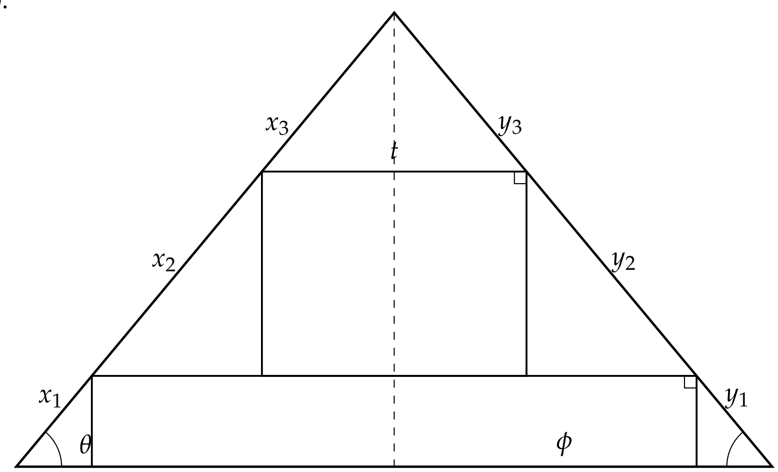
Therefore $Q(x) > 0$ for all x.

39. Let the dimensions of the cube be $a, b, c > 0$ so that we have $a \cdot b \cdot c = V$. The surface area $ab + bc + ca$. Invoking AM-GM inequality

$$\frac{ab + bc + ca}{3} \geq (a^2b^2c^2)^{\frac{1}{3}} = V^{\frac{2}{3}}.$$

For the surface area to be minimum $ab = bc = ca \implies a = b = c$.

40.



Let $A(M)$ denote the remaining area. Therefore:

$$A(M) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\sin(2\theta) + \frac{1}{2}(y_1^2 + y_2^2 + y_3^2)\sin(2\phi)$$

$$A(T) = \frac{1}{2}(x_1 + x_2 + x_3)^2\sin(2\theta) + \frac{1}{2}(y_1 + y_2 + y_3)^2\sin(2\phi)$$

The objective is to maximise $\frac{A(R)+A(S)}{A(T)} = 1 - \frac{A(M)}{A(T)}$, which is as good as minimising $\frac{A(M)}{A(T)}$. Using Cauchy-Schwarz inequality

$$(1^2 + 1^2 + 1^2)(x_1^2 + x_2^2 + x_3^2) \geq (x_1 + x_2 + x_3)^2$$

Since θ is an acute angle, so $\sin 2\theta > 0$ and hence the above inequality can be restated as

$$\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\sin(2\theta) \geq \frac{1}{6}(x_1 + x_2 + x_3)^2\sin(2\theta)$$

Similarly it can be shown that

$$\frac{1}{2}(y_1^2+y_2^2+y_3^2)\sin(2\theta) \geq \frac{1}{6}(y_1+y_2+y_3)^2\sin(2\theta)$$

Adding these two inequalities we get:

$$A(M) \geq \frac{1}{3}A(T) \implies \frac{A(M)}{A(T)} \geq \frac{1}{3}.$$

So, the maximum value of $\frac{A(R)+A(S)}{A(T)}$ is $\frac{2}{3}$. This equality is achieved when $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$. In fact there is nothing special about this result for n rectangles we can analogously argue that the result will be $\frac{n-1}{n}$.

41. Since $f''(x) - 5f'(x) + 6f(x) \geq 0$

$$f''(x) - 2f'(x) \geq 3(f'(x) - 2f(x)) \tag{A}$$

Say $f'(x) - 2f(x) = g(x)$, then (A) transforms into

$$g'(x) \geq 3g(x) \implies g'(x) - 3g(x) \geq 0.$$

$$e^{-3x}g'(x) - e^{-3x}g(x) \geq 0 \implies (e^{-3x}g(x))' \geq 0$$

So the function $e^{-3x}g(x)$ is increasing. Therefore, for $x > 0$, we have $e^{-3x}g(x) \geq e^{-0}g(0) = -2$. Hence $g(x) \geq -2e^{3x}$.

$$\implies f'(x) - 2f(x) \geq -2e^{3x} \implies e^{-2x}f'(x) - 2e^{-2x}f(x) \geq -2e^{x}$$

$$\implies (e^{-2x}f(x))' \geq -2e^{x} = (-2e^{x})'$$

$$\implies (e^{-2x}f(x) + 2e^{x})' \geq 0$$

So the function $e^{-2x}f(x) + 2e^{x}$ is increasing. So for $x > 0$

$$e^{-2x}f(x) + 2e^{x} \geq f(0) + 2 = 3$$

$$\implies f(x) \geq 3e^{2x} - 2e^{3x}.$$

42. Since P has only real zeros, we can write

$$P(x) = a\prod_{i=1}^{n}(x - r_i),$$

where $a \neq 0$ is a real constant and $r_1, r_2, \ldots, r_n$ are real numbers.

We first consider the case when $P(x) \neq 0$. Then, using logarithmic differentiation we have

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n}\frac{1}{x - r_i}.$$

Let

$$S = \sum_{i=1}^{n} \frac{1}{x - r_i}, \quad \text{so that} \quad P'(x) = P(x)S.$$

Differentiating again,

$$\frac{P''(x)}{P(x)} = \left(\frac{P'(x)}{P(x)}\right)^2 - \sum_{i=1}^{n} \frac{1}{(x - r_i)^2} = S^2 - T,$$

where

$$T = \sum_{i=1}^{n} \frac{1}{(x - r_i)^2}.$$

Thus,

$$P''(x) = P(x)(S^2 - T).$$

The inequality to prove becomes

$$(n-1)(P'(x))^2 \geq nP(x)P''(x).$$

Substituting the expressions for $P'(x)$ and $P''(x)$, we get

$$(n-1)P(x)^2 S^2 \geq nP(x)^2(S^2 - T).$$

Since $P(x) \neq 0$, we can divide both sides by $P(x)^2$ to obtain

$$(n-1)S^2 \geq n(S^2 - T),$$

which simplifies to

$$(n-1)S^2 \geq nS^2 - nT \quad \Rightarrow \quad nT \geq S^2.$$

So it suffices to prove that

$$S^2 \leq nT.$$

But by the Cauchy Schwarz inequality,

$$\left(\sum_{i=1}^{n} 1 \cdot \frac{1}{x - r_i}\right)^2 \leq \left(\sum_{i=1}^{n} 1^2\right) \left(\sum_{i=1}^{n} \left(\frac{1}{x - r_i}\right)^2\right) = nT.$$

Hence, $S^2 \leq nT$, and the inequality is proved for $P(x) \neq 0$.

Now consider the case when $P(x) = 0$. Then the inequality becomes

$$(n-1)(P'(x))^2 \geq 0,$$

which is clearly true. If x is a simple zero, then $P'(x) \neq 0$, so the inequality is strict. If x is a multiple zero, then $P'(x) = 0$, and both sides are zero, so equality holds.

We now examine the conditions for equality. For $P(x) \neq 0$, equality holds if and only if $S^2 = nT$. By the Cauchy Schwarz inequality, this occurs if and only if the numbers $\frac{1}{x-r_i}$ are all equal. That is, there exists a constant λ such that for all i,

$$\frac{1}{x - r_i} = \lambda.$$

This implies that all $x - r_i$ are equal, so $r_i = x - c$ for some constant c independent of i. Hence, all zeros of P are equal, and $P(x) = a(x-c)^n$. In this case, one can verify directly that equality holds for all x.

Also, if $P(x) = 0$ and $P'(x) = 0$ (i.e., x is a multiple root), then equality holds.

Therefore, equality holds if and only if either all zeros of P are equal (so that P is a constant multiple of $(x-c)^n$) or x is a multiple root of P.

43. Let $g(x) = \frac{1}{2}f^2(x) + f'(x)$. Define $g(0) = 0$. It is sufficient to prove that for some $0 < t \leq 1$ we have $g(t) = 0$.

1. If f is never zero, let

   $$h(x) = \frac{x}{2} - \frac{1}{f(x)}$$

   Since $h(1) = h(0) = \frac{-1}{2}$, there exists a real number $0 < t < 1$ for which $h'(t) = 0$. But $g = f^2 \cdot h'$, and we are done.

2. If f has at least one zero, let t_1 be the first zero and t_2 be the last zero. So we have $0 < t_1 \leq t_2 < 1$. The function f is positive on the intervals $[0, t_1)$ and $(t_2, 1]$. Therefore, $f'(t_1) \leq 0$ and $f'(t_2) \geq 0$. And hence $g(t_1) = f'(t_1) \leq 0$ and $g(t_2) = f'(t_2) \geq 0$. Therefore, there exists a $t \in [t_1, t_2]$ such that $g(t) = 0$.

Comment how did we magically arrive upon the function $h(x) = \frac{x}{2} - \frac{1}{f(x)}$? Well because we wanted to show $g(x)$ zero one more time. So we just assumed $g(x) = \frac{1}{2}f^2(x) + f'(x) = 0$ or $\frac{f'(x)}{f^2(x)} = \frac{-1}{2}$. Integrating this we get $\frac{x}{2} - \frac{1}{f(x)}$.

44. The given equation is:

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

Replacing x by $\frac{a}{x}$ that is $x \to \frac{a}{x}$

$$f'(x) = \frac{a}{xf(\frac{a}{x})}$$

Differentiating this with respect to x, we get

$$f''(x) = -\frac{a}{x^2 f(\frac{a}{x})} + \frac{a^2 f'(\frac{a}{x})}{x^3 f(\frac{a}{x})^2}$$

Substituting from the original equation we get:

$$f''(x) = -\frac{f'(x)}{x} + \frac{f'(x)^2}{f(x)}$$

$$\implies xf(x)f''(x) + f(x)f'(x) = xf'(x)^2$$

Dividing by $f(x)^2$ and rearranging give us:

$$\frac{d}{dx}\left(\frac{xf'(x)}{f(x)}\right) = 0$$

$$\implies \frac{f'(x)}{f(x)} = \frac{d}{x}, \text{ for some constant } d$$

$$\implies f(x) = cx^d, \text{ as desired.}$$

45. If the function any one of them vanishes then we are done. So, we just need to show that there's there is some a for which

$$f(a)f'(a)f''(a)f'''(x) > 0$$

Say that this is not possible, then we have we must have either of the two

1. either exactly one of these three quantities is always negative and the others are always positive or

2. three are always negative and one is always positive(this is the same as above just replace f by $-f$).

Say that $f'' < 0$ for all x. Since $f'''(x) > 0$, therefore $f''(x)$ is strictly increasing. And hence $f''(x) < f''(0)$ for $x > 0$.

46. If at least one of $f(a)$, $f'(a)$, $f''(a)$, or $f'''(a)$ vanishes at some point a, then we are done. Otherwise, by the Intermediate Value Theorem, each of $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ is either strictly positive or strictly negative on the real line. Thus, it suffices to show that their product is positive for a single value of x.

By replacing $f(x)$ with $-f(x)$ if necessary, we may assume that $f''(x) > 0$; and by replacing $f(x)$ with $f(-x)$ if necessary, we may assume that $f'''(x) > 0$. Notice that these substitutions do not alter the sign of the product

$$f(x)f'(x)f''(x)f'''(x).$$

Since $f'''(x) > 0$, it follows that $f''(x)$ is increasing. Thus, for $a > 0$,

$$f'(a) = f'(0) + \int_0^a f''(t)\,dt \;\; \geq \;\; f'(0) + af''(0).$$

In particular, $f'(a) > 0$ for sufficiently large a. Similarly, since $f''(x) > 0$, we have that $f(a)$ is positive for large a. Therefore,

$$f(x)f'(x)f''(x)f'''(x) > 0$$

for sufficiently large x.

46. Since $f'(x) > 0$, the function f is increasing. In particular, for $x \leq 0$ we have

$$f(x) \leq f(0).$$

Given $f'''(x) \leq f(x)$, it follows that

$$f^{(3)}(x) \leq f(0), \qquad x \leq 0.$$

Integrating from x to 0 yields

$$f''(x) \geq f''(0) + f(0)x, \qquad x \leq 0.$$

Integrating once more gives

$$f'(x) \geq f'(0) + f''(0)x + \tfrac{1}{2}f(0)x^2, \qquad x \leq 0.$$

Thus the quadratic

$$Q_1(x) = f'(0) + f''(0)x + \tfrac{1}{2}f(0)x^2$$

must remain positive for all $x \leq 0$. Since its coefficients are all positive, it cannot vanish for $x \geq 0$ either. Hence $Q_1(x)$ has no real root. This forces its discriminant to be negative:

$$f''(0)^2 - 2f(0)f'(0) < 0.$$

Now observe that $f'''(x) > 0$ implies f'' is increasing. For $x \leq 0$ we then have

$$f''(x) \leq f''(0),$$

and integrating gives

$$f'(x) \geq f'(0) + f''(0)x, \qquad x \leq 0,$$

$$f(x) \leq f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2, \qquad x \leq 0.$$

Thus the quadratic

$$Q_2(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2$$

must also have no real roots, forcing its discriminant to be negative:

$$f'(0)^2 - 2f(0)f''(0) < 0.$$

Putting these two inequalities together, we obtain

$$f'(0)^4 < 4f(0)^2 f''(0)^2 < 8f(0)^3 f'(0).$$

Dividing through by $f(0)^3 > 0$, we conclude

$$f'(0) < 2f(0).$$

Since the choice of the base point 0 was arbitrary, the inequality

$$f'(x) < 2f(x)$$

holds for all real x.

47. We begin by considering the case when f is a polynomial.

Case 1. Constant polynomial. Suppose $f(x) \equiv c$ for some constant $c \in \mathbb{R}$. The condition $f(f(x)) = f(x)^{2013}$ becomes

$$c = c^{2013}.$$

Equivalently,

$$c(c^{2012} - 1) = 0,$$

so the possible values are $c = 0$, $c = 1$, or $c = -1$. Thus, there are exactly three constant solutions.

Case 2. Non-constant polynomial. If f is non-constant, consider its range

$$A = \{f(x) : x \in \mathbb{R}\}.$$

For each $a \in A$, the condition gives

$$f(a) = a^{2013}.$$

Hence every element of A is a root of the polynomial $f(x) - x^{2013}$. But since f is continuous and non-constant, A is infinite, implying that $f(x) - x^{2013}$ has infinitely many roots. Therefore, it must be the zero polynomial, i.e.

$$f(x) = x^{2013}, \qquad \forall x \in \mathbb{R}.$$

Thus, the only polynomial solutions are the three constant functions $f(x) = 0, 1, -1$ and the non-constant polynomial $f(x) = x^{2013}$. In total, there are exactly four polynomial solutions.

Infinitely many non-polynomial solutions. To construct infinitely many other functions, observe that the condition is satisfied if

$$f(0) = 0, \quad f(1) = 1, \quad f(-1) = -1,$$

and for all other real numbers x, the value $f(x)$ is chosen arbitrarily from the set $\{0, 1, -1\}$. It is straightforward to check that any such definition ensures $f(f(x)) = f(x)^{2013}$. Since there are infinitely many ways to assign values in this manner, we obtain infinitely many valid (non-polynomial) solutions.

Another systematic construction is as follows: Fix any real number a. Define

$$S = \{a^{2013^i} : i = 0, 1, 2, \dots\}.$$

For all $x \in S$, set $f(x) = x^{2013}$, which is consistent with the functional equation. For $x \notin S$, assign $f(x)$ to be any element of S. This again provides infinitely many valid functions.

49. Since we have

$$f(x^2 - y^2) = xf(x) - yf(y)$$

Put $x = y = 0$ to get $f(0) = 0$. In addition, put $y = 0$ in the original functional equation to get:

$$f(x^2) = xf(x) \tag{A}$$

Define $f(x) - cx = g(x)$, for some constant c. Equation (A) transforms into

$$g(x^2) + cx^2 = x(g(x) + cx)$$

$$\implies g(x^2) = xg(x) \tag{B}$$

Also, notice that $g(0) = f(0) = 0$. Since equations (A) and (B) are analogous and $f(0) = g(0) = 0$, therefore any solution of equation (A) must be a scalar multiple of the solution of equation (B). So, we have $f(x) = \lambda g(x)$ for some scalar λ. Putting this in equation (A), we get $f(x) = kx$, where k is some constant. And it can be seen that this function satisfies the original functional equation.

50. Say that $x = a, y = a + d, z = a + 2d, t = a + 3d$. Now we have the following condition

$$f(a) + f(a + 3d) = f(a + d) + f(a + 2d)$$

And similarly once can argue

$$f(a - d) + f(a + 2d) = f(a) + f(a + d)$$

Adding these equations, we get

$$f(a - d) + f(a + 3d) = 2f(a + d)$$

$$\implies \frac{f(a-d)+f(a+3d)}{2} = f\left(\frac{(a-d)+(a+3d)}{2}\right)$$

That is, Jensen's functional equation. And now we can take it from here.

51. Say that $x-y=a$ and $x+y=b$. So, the given functional equation converts into

$$af(b)-bf(a)=(b^2-a^2)ab$$

For $ab \neq 0$, we have

$$\frac{f(b)}{b}-\frac{f(a)}{a}=b^2-a^2$$

$$\implies \frac{f(b)}{b}-b^2=\frac{f(a)}{a}-a^2$$

$$\implies \frac{f(x)}{x}-x^2=c, \text{ where } c \text{ is some constant.}$$

$$\implies f(x)=cx+x^3$$

Integrals

1.

$$\text{Let, } I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$$

$$\implies I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}} dx$$

$$2I = \int_{-\pi}^{\pi} \cos^2 x dx = \pi \implies I = \frac{\pi}{2}.$$

2. Let $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1+\cos x} dx$

$$\implies I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1-\cos x} dx$$

$$\implies 2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x} \right) dx$$

$$\implies I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc^2 x \, dx = 2.$$

3.

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} \, dx = \int_0^1 \frac{x^4((x^2+1)^2 - 4x(1+x^2) + 4(x^2-1)(x^2+1) + 4)}{1+x^2} \, dx$$

$$= \frac{22}{7} - \pi.$$

4.

$$\text{Say, } I_n = \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} \, dx$$

$$\implies I_n = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)} \, dx = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} \, dx$$

$$2I_n = 2\pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} \, dx$$

$$I_n = 4\pi \int_0^{\frac{\pi}{2}} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} \, dx$$

$$I_n = 4\pi \int_0^{\frac{\pi}{2}} \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x} \, dx$$

$$\implies 2I_n = 4\pi \int_0^{\frac{\pi}{2}} 1 \, dx = 2\pi^2$$

$$I_n = \pi^2.$$

5.
$$\begin{aligned}\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)\,dx &= \int_0^1 \tan^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right)\,dx\\ &= \int_0^1 \tan^{-1}(x)\,dx - \int_0^1 \tan^{-1}(x-1)\,dx\\ &= \int_0^1 \tan^{-1}(x)\,dx - \int_0^1 \tan^{-1}((1-x)-1)\,dx\\ &== \int_0^1 \tan^{-1}(x)\,dx + \int_0^1 \tan^{-1}(x)\,dx = 2\int_0^1 \tan^{-1} x\,dx.\end{aligned}$$

9. Say that $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x and $\{x\}$ is its fractional part.

$$\begin{aligned}\int_1^x \lfloor x \rfloor(\lfloor x \rfloor+1)f(u)du &= \sum_{k=1}^{\lfloor x \rfloor-1} k(k+1)\int_k^{k+1} f(u)du + \lfloor x \rfloor(\lfloor x \rfloor+1)\int_{\lfloor x \rfloor}^x f(u)du\\ &= \sum_{k=1}^{\lfloor x \rfloor} k(k+1)\int_k^x f(u)du\\ &= 2\sum_{k=1}^{\lfloor x \rfloor} k\int_k^x f(u)du.\end{aligned}$$

Or it is easy to see that $n(n+1) = 2\sum_{i=1}^{n} i$

$$\begin{aligned}\int_1^x \lfloor x \rfloor(\lfloor x \rfloor+1)f(u)du &= \int_1^x \sum_{i=1}^{\lfloor x \rfloor} 2if(u)du\\ &= 2\sum_{i=1}^{\lfloor x \rfloor} i\int_i^x f(u)du.\end{aligned}$$

11. In the integral $\int_{-100}^{-10} \frac{(x^2-x)^2}{(x^3-3x+1)^2}\,dx$ put $y = \frac{x-1}{x}$ to obtain $\int_{\frac{101}{100}}^{\frac{11}{10}} \frac{y^2}{(y^3-3y+1)^2}dy$. Similarly in the integral $\int_{\frac{1}{101}}^{\frac{1}{11}} \frac{(x^2-x)^2}{(x^3-3x+1)^2}\,dx$ put $y = \frac{1}{1-x}$ to get the integral as $\int_{\frac{101}{100}}^{\frac{11}{10}} \frac{(y^2-1)^2}{(y^3-3y+1)^2}dy$. Put them all together to get:

$$\begin{aligned}&\int_{\frac{1}{101}}^{\frac{11}{10}} \left(\frac{t^2-t}{t^3-3t+1}\right)^2\left(1+\frac{1}{t^2}+\frac{1}{(1-t)^2}\right)dt\\ &-\frac{x^2-x}{x^3-3x+1}\Bigg|_{\frac{1}{101}}^{\frac{11}{10}}\end{aligned}$$

which is rational.

12. The series on the left is the Taylor expansion of $xe^{-x^2/2}$. Since the terms of the second sum are nonnegative, we may interchange the summation and integration by the Monotone Convergence Theorem. Thus, the given expression becomes

$$\sum_{n=0}^{\infty} \int_0^{\infty} x^{2n+1} e^{-x^2/2} \frac{dx}{2^n (2n)!}.$$

To evaluate these integrals, observe first that

$$\int_0^{\infty} xe^{-x^2/2}\, dx = \Big[-e^{-x^2/2} \Big]_0^{\infty} = 1.$$

By integration by parts, we have

$$\int_0^{\infty} x^{2n} (xe^{-x^2/2})\, dx = -x^{2n} e^{-x^2/2} \Big|_0^{\infty} + \int_0^{\infty} 2nx^{2n-1} e^{-x^2/2}\, dx.$$

$$\Rightarrow \quad \int_0^{\infty} x^{2n+1} e^{-x^2/2}\, dx = 2n \int_0^{\infty} x^{2n-1} e^{-x^2/2}\, dx.$$

Hence, by induction,

$$\int_0^{\infty} x^{2n+1} e^{-x^2/2}\, dx = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!.$$

Substituting this result back, the desired integral becomes

$$\sum_{n=0}^{\infty} \frac{1}{2^n n!} = \sqrt{e}.$$

14. Notice that

$$\int \left(\Pi_{r=1}^{n}(x+r)\right) \left(\sum_{k=1}^{n} \frac{1}{x+k} \right) dx = \left(\Pi_{r=1}^{n}(x+r)\right)$$

16. Use integration by parts or one can also try induction.

17. Differentiating with respect to x, we get

$$2f(x)f'(x) = f(x) \frac{\sin x}{2 + \cos x}$$

Either $f(x) = 0$ or

$$f'(x) = \frac{1}{2} \frac{\sin x}{2 + \cos x}$$

$$\Longrightarrow f(x) = \frac{1}{2} \log(2 + \cos x) + C$$

Notice $f(0) = 0$, that gives us the value of C.

18. (a) Notice that for $x > 0$, we have $e^x > 1$. Also notice that $\int_1^x \frac{1}{t}\, dt = \log x$.

$$\text{For } x \geq 1, \int_1^x \frac{e^t}{t} dt \geq \int_1^x \frac{1}{t}\, dt = \log x.$$

And for $0 < x < 1$

$$\int_x^1 \frac{e^t}{t} dt \geq \int_x^1 \frac{1}{t}\, dt \implies \int_1^x \frac{e^t}{t} dt \leq \int_1^x \frac{1}{t}\, dt.$$

(b) Put $t + a = u$ to get the result.

19.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{x+1} dx$$

Put $x = 2t$

$$\implies \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{x+1} dx = \int_0^{\pi} \frac{\sin x}{x+2}\, dx$$

Now integrating by parts $u = \frac{1}{x+2}$ and $dv = \sin x dx$. So we have $du = -\frac{1}{(x+2)^2} dx$ and $v = -\cos x$. So, the given integral becomes

$$\left. \frac{-\cos x}{x+1} \right|_0^{\pi} + A.$$

22. Define

$$\Phi(t) := \int_0^t f(x)\, dx \qquad (0 \leq t \leq 1).$$

Then Φ is continuous and $\Phi(0) = \Phi(1) = 0$. Now define

$$F(c) := \int_0^c x f(x)\, dx \qquad (0 \leq c \leq 1).$$

We will show that there exists $c \in (0,1)$ such that $F(c) = 0$.

If $\Phi \equiv 0$ on $[0,1]$, then $f \equiv 0$ and hence $F(c) = 0$ for every $c \in (0,1)$, so the claim is true. Assume now that Φ is not identically zero. Since $\Phi(0) = \Phi(1) = 0$, Φ must attain either a positive maximum or a negative minimum at some interior point.

Case 1. Suppose Φ attains a positive maximum $M > 0$ at some $a \in (0,1)$. Integrating by parts,

$$F(a) = \int_0^a x f(x)\, dx = a\Phi(a) - \int_0^a \Phi(x)\, dx = aM - \int_0^a \Phi(x)\, dx.$$

For $x \in [0,a]$, we have $\Phi(x) \leq M$, hence

$$\int_0^a \Phi(x)\,dx \leq aM,$$

with equality only if $\Phi(x) \equiv M$ for all $x \in [0,a]$, which is impossible since $\Phi(0) = 0$. Therefore, $\int_0^a \Phi(x)\,dx < aM$, and hence $F(a) > 0$.

Case 2. Suppose Φ attains a negative minimum $m < 0$ at some $b \in (0,1)$. Then

$$F(b) = b\Phi(b) - \int_0^b \Phi(x)\,dx = bm - \int_0^b \Phi(x)\,dx.$$

Since $\Phi(x) \geq m$ for all $x \in [0,b]$ and $\Phi(0) = 0 \neq m$, we have

$$\int_0^b \Phi(x)\,dx > bm,$$

and thus $F(b) < 0$.

In either case, we have found points $a, b \in (0,1)$ such that $F(a) > 0$ and $F(b) < 0$. By the continuity of F, there exists $c \in (0,1)$ such that $F(c) = 0$.

$$\boxed{\text{Hence, there exists } c \in (0,1) \text{ such that } \int_0^c x f(x)\,dx = 0.}$$

$\square$

23. Differentiating with respect to b keeping a as constant

$$(a+2b)\int_a^b f(x)dx = f(b)(2b^2 - a^2 - ab)$$

Put $a = 0$ to get

$$\int_0^b f(x)dx = bf(b)$$

Differentiating again with respect to b to get $f'(b) = 0$. Which means that $f(b)$ is constant. Putting back in the original equation we get $f(x) = 1$ and we can see that it satisfies the original equation.

24. For $\frac{1}{2} \leq x \leq 1$, we have

$$8 \leq f'(x) \leq 96 \implies \int_{\frac{1}{2}}^x 8\,dx \leq \int_{\frac{1}{2}}^x f'(x)\,dx \leq \int_{\frac{1}{2}}^x 96\,dx$$

$$8x - 4 \leq f(x) \leq (96x - 48)$$

$$\int_{\frac{1}{2}}^1 (8x-4)\,dx \leq \int_{\frac{1}{2}}^1 f(x)\,dx \leq \int_{\frac{1}{2}}^1 (96x-48)\,dx$$

$$1 \leq \int_{\frac{1}{2}}^{1} f(x)\, dx \leq 12.$$

26. So for $0 < x < 1$ we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \leq 1 - x^2 + x^4$$

And further $0 < 1 - x^2 + x^4 < 1$

$$\frac{1}{1+a^2}\int_0^a (1-x^2+x^4)\, dx \leq \int_0^a \frac{1-x^2+x^4}{1+x^2}\, dx \leq \int_0^a \frac{1}{1+x^2}\, dx \leq \int_0^a (1-x^2+x^4)\, dx$$

And the result follows.

28. Since for $0 \leq x \leq 1$, we have

$$\frac{1}{\sqrt{4-x^2}} \leq \frac{1}{\sqrt{4-x^2-x^3}} \leq \frac{1}{\sqrt{4-2x^2}}$$

$$\implies \int_0^1 \frac{1}{\sqrt{4-x^2}}\, dx \leq \int_0^1 \frac{1}{\sqrt{4-x^2-x^3}}\, dx \leq \int_0^1 \frac{1}{\sqrt{4-2x^2}}\, dx$$

And the result follows.

29. Notice that

$$\left|\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{1+x}\, dx\right| - \left|\int_{(n+1)\pi}^{(n+2)\pi} \frac{\sin x}{1+x}\, dx\right| > 0$$

In the latter integral replace $x - \pi = t \implies dx = dt$.

$$\implies \left|\int_{(n+1)\pi}^{(n+2)\pi} \frac{\sin x}{1+x}\, dx\right| = \left|\int_{n\pi}^{(n+1)\pi} \frac{\sin(x+\pi)}{1+\pi+x}\, dx\right|$$

$$= \left|\int_{n\pi}^{(n+1)\pi} \frac{\sin x}{1+\pi+x}\, dx\right|$$

Now $I_1 = \int_0^{\pi} \frac{\sin x}{1+x}\, dx$ and $I_2 = I_1 + \int_{\pi}^{2\pi} \frac{\sin x}{1+x}\, dx$ and $I_3 = I_2 + \int_{2\pi}^{3\pi} \frac{\sin x}{1+x}\, dx$ and $I_4 = I_3 + \int_{3\pi}^{4\pi} \frac{\sin x}{1+x}\, dx$. Based on the sign of $\sin x$ between $(n\pi, (n+1)\pi)$, we have

$$I_1 > I_3 > I_4 > I_2.$$

30. Notice that

$$I_n \cdot I_{n+2} = \left(\int_a^b \frac{(f(x))^{n+1}}{(g(x))^n} dx\right) \cdot \left(\int_a^b \frac{(f(x))^{n+3}}{(g(x))^{n+2}} dx\right)$$

$$\geq \left(\int_a^b \frac{(f(x))^{n+2}}{(g(x))^{n+1}} dx\right)^2 = I_{n+1}^2 \text{ (Cauchy-Schwarz for integrals).}$$

Therefore, the sequence $\left\{\frac{I_{n+1}}{I_n}\right\}_{n=0}^{\infty}$ is an increasing sequence and $\frac{I_1}{I_0} > 1$. And hence the result follows.

31. By Hölder's inequality we have

$$\int_0^3 f(x)\cdot 1dx \le \left(\int_0^3 |f(x)|^3 dx\right)^{\frac{1}{3}} \left(\int_0^3 1^{\frac{3}{2}} dx\right)^{\frac{2}{3}} = 3^{\frac{2}{3}} \left(\int_0^3 |f(x)|^3 dx\right)^{\frac{1}{3}}$$

$$\implies \frac{\left(\int_0^3 f(x)dx\right)^3}{\int_0^3 f(x)^3 dx} \le 9.$$

The equality is attained when f is constant.

32. This is the same as problem number 32 of the solved section. Why?

33. Obviously $f(x) = \int_a^x f'(t)dt$. Squaring both sides and invoking Cauchy-Schwarz inequality we get

$$f(x)^2 = \left(\int_a^b f'(t)dt\right)^2 \le (b-a)\int_a^b f'(t)^2 dt$$

Integrating with respect to x

$$\int_a^b f(x)^2 dx \le (b-a)^2 \int_a^b f'(t)^2 dt.$$

34. Since f is continuous and bijective, therefore it must either be strictly increasing or strictly decreasing. Since $f(0) = 0$, therefore f is strictly increasing and $f(1) = 1$. Using Young's inequality we have

$$\int_0^x f(t)dt + \int_0^x f^{-1}(t)dt \ge x^2,\ x \in [0,1]$$

$$\implies \int_0^x (f(t) + f^{-1}(t) - 2t)dt = 0,\ x \in [0,1]$$

Let $G : [0,1] \to \mathbb{R}$ with $G(x) = \int_0^x (f(t) + f^{-1}(t) - 2t)dt$, then $G(1) = G(0) = 0$, $G(x) \ge 0\ \in [0,1]$.

$$\int_0^1 x^\alpha (f(x) + f^{-1}(x))\, dx = 2\int_0^1 x^{\alpha+1} dx + \int_0^1 x^\alpha (f(x) + f^{-1}(x) - 2x)dx$$

$$= \frac{2}{2+\alpha} + \int_0^1 x^\alpha G'(x)dx = \frac{2}{2+\alpha} + \lim_{a\to 0}(G(1) - G(a)\cdot a^\alpha - \int_a^1 \alpha \cdot x^{\alpha-1} G(x)dx)$$

$$\le \frac{2}{\alpha+2}.$$

And the result follows.

35. From the hypothesis,

$$\int_x^1 f(t)\,dt \geq \frac{1-x^2}{2}, \quad \forall x \in [0,1].$$

Integrate both sides with respect to x from 0 to 1. Swapping the order of integration on the left-hand side (justified by Fubini's theorem, since f is continuous), we have

$$\int_0^1 \left(\int_x^1 f(t)\,dt \right) dx \geq \int_0^1 \frac{1-x^2}{2}\,dx.$$

The left-hand side simplifies as

$$\int_0^1 \int_x^1 f(t)\,dt\,dx = \int_0^1 \int_0^t f(t)\,dx\,dt = \int_0^1 t\,f(t)\,dt,$$

and the right-hand side is

$$\int_0^1 \frac{1-x^2}{2}\,dx = \left[\frac{x}{2} - \frac{x^3}{6} \right]_0^1 = \frac{1}{3}.$$

Thus, we obtain the useful inequality

$$\int_0^1 t\,f(t)\,dt \geq \frac{1}{3}.$$

Now, applying the Cauchy–Schwarz inequality to the functions $f(t)$ and t, we get

$$\left(\int_0^1 t\,f(t)\,dt \right)^2 \leq \left(\int_0^1 f^2(t)\,dt \right) \left(\int_0^1 t^2\,dt \right).$$

Since $\displaystyle\int_0^1 t^2\,dt = \frac{1}{3}$ and $\displaystyle\int_0^1 t\,f(t)\,dt \geq \frac{1}{3}$, it follows that

$$\left(\frac{1}{3} \right)^2 \leq \left(\int_0^1 f^2(t)\,dt \right) \left(\frac{1}{3} \right),$$

which simplifies to

$$\int_0^1 f^2(t)\,dt \geq \frac{1}{3}.$$

36. $\int_0^1 (1-x^{50})^{101}\,dx = \int_0^1 (1-x^{50})^{100}(1-x^{50})\,dx$

$$= \int_0^1 (1-x^{50})^{100}\,dx - \int_0^1 x^{50}(1-x^{50})^{100}dx$$

Integrating the latter integral by parts
$u = x, dv = x^{49}(1-x^{50})^{100}dx \implies du = dx$ and $v = -\frac{1}{5050}(1-x^{50})^{101}$ So the latter integral is equal to

$$\left[\frac{-1}{5050}x(1-x^{50})^{100}\right]_0^1 + \int_0^1 \frac{1}{5050}(1-x^{50})^{101}dx$$

$$= \int_0^1 \frac{1}{5050}(1-x^{50})^{101}dx$$

$$\implies \int_0^1 (1-x^{50})^{101}\,dx = \int_0^1 (1-x^{50})^{100}\,dx - \int_0^1 \frac{1}{5050}(1-x^{50})^{101}dx$$

That gives us the required ratio.

43. Claim: The given integral is 0 for numbers of the form $1, 2 \equiv (\mod 4)$ and non-zero for integers of the form $0, 3 \equiv (\mod 4)$. The given integral can be written as

$$\int_0^{2\pi} \cos x \cos 2x \cdots \cos mx dx = \int_0^{\pi} \cos x \cos 2x \cdots \cos mx dx + \int_{\pi}^{2\pi} \cos x \cos 2x \cdots \cos mx dx$$

For the second integral, say $x - \pi = t \implies dx = dt$. So, the original integral transforms into

$$= \int_0^{\pi} \cos x \cos 2x \cdots \cos mx dx + \int_0^{\pi} \cos(\pi + x)\cdot\cos(2\pi + 2x)\cdots\cos(m\pi + mx)dx$$

Notice that $\cos(4n\pi + 4nx) = \cos(4nx)$, $\cos((4n+1)\pi + (4n+1)x) = -\cos(4n+1)x$, $\cos((4n+2)\pi + (4n+2)x) = \cos(4n+2)x$ and finally $\cos((4n+3)\pi + (4n+3)x) = -\cos(4n+3)x$. Say that $m \equiv 1(\mod 4)$, that is, $m = 4n+1$. Similarly, find the case where $m \equiv 2(\mod 4)$. Now, the integral becomes

$$\int_0^{\pi} \cos x \cdots \cos(4n+1)x dx + \int_0^{\pi} \cos(\pi + x)\cdots\cdot\cos((4n+1)\pi + (4n+1)x)dx$$

$$= \int_0^{\pi} \cos x \cdots \cos(4n+1)x dx - \int_0^{\pi} \cos x \cdots \cos(4n+1)x dx = 0.$$

For $n \equiv 0(\mod 4)$, the given integral is equal to

$$2\int_0^{\pi} \cos x \cdot \cos 2x \cdots \cos(4nx)\, dx$$

Now put $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ to get the result.

44. Let

$$f(r) = \int_0^{\pi/2} x^r \sin x\, dx.$$

Since $0 \le \sin x \le 1$,

$$f(r) < \int_0^{\pi/2} x^r\,dx = \frac{(\pi/2)^{r+1}}{r+1}.$$

Also, since $\sin x \ge \frac{2x}{\pi}$ for $0 \le x \le \pi/2$,

$$f(r) > \int_0^{\pi/2} \frac{2x^{r+1}}{\pi}\,dx = \frac{(\pi/2)^{r+1}}{r+2}.$$

Hence

$$\lim_{r\to\infty} r\left(\frac{2}{\pi}\right)^{r+1} f(r) = 1.$$

Now, by integration by parts,

$$\int_0^{\pi/2} x^r \cos x\,dx = \frac{1}{r+1}\int_0^{\pi/2} x^{r+1}\sin x\,dx = \frac{f(r+1)}{r+1}.$$

Therefore,

$$\frac{\int_0^{\pi/2} x^r \sin x\,dx}{\int_0^{\pi/2} x^r \cos x\,dx} = (r+1)\frac{f(r)}{f(r+1)}.$$

Using the previous limit,

$$\lim_{r\to\infty}\frac{f(r)}{f(r+1)} = \lim_{r\to\infty}\frac{r(2/\pi)^{r+1}f(r)}{(r+1)(2/\pi)^{r+2}f(r+1)} = \frac{2}{\pi}.$$

Thus,

$$\lim_{r\to\infty} r^{-1}\cdot(r+1)\frac{f(r)}{f(r+1)} = \frac{2}{\pi}.$$

Setting $c = -1$ gives the finite positive limit

$$L = \frac{2}{\pi}.$$

$$\boxed{(c, L) = \left(-1, \frac{2}{\pi}\right).}$$

45. Say $I_n = \int_0^\pi \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$. Obviously $I_0 = \pi$. Consider

$$I_{n+1} - I_n = \int_0^\pi \frac{\sin(n+\frac{3}{2})t - \sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

$$= 2\int_0^\pi \cos(n+2)t\,dt = 0.$$

Therefore, the sequence I_n is a constant.

46.

$$I(\lambda) = \int_0^\pi \sin\Big((\lambda + \tfrac{1}{2})t\Big)\, t\Big(\frac{2}{t} - \frac{1}{\sin(t/2)}\Big)\, dt = \int_0^\pi \sin\Big((\lambda + \tfrac{1}{2})t\Big)\, g(t)\, dt,$$

where

$$g(t) := 2 - \frac{t}{\sin(t/2)}.$$

First, note that as $t \to 0$,

$$\sin\frac{t}{2} = \frac{t}{2} - \frac{t^3}{48} + \cdots,$$

so

$$\frac{t}{\sin(t/2)} = 2\left(1 + \frac{t^2}{24} + \cdots\right) = 2 + \frac{t^2}{12} + \cdots.$$

Hence

$$g(t) = 2 - \frac{t}{\sin(t/2)} \sim -\frac{t^2}{12} \to 0,$$

which means g extends continuously to $t = 0$ with $g(0) = 0$. Moreover, g is C^1 on $[0, \pi]$, so g and g' are bounded.

We now integrate by parts:

$$u = g(t), \qquad dv = \sin\Big((\lambda + \tfrac{1}{2})t\Big)\, dt.$$

Then

$$v = -\frac{\cos\Big((\lambda + \frac{1}{2})t\Big)}{\lambda + \frac{1}{2}},$$

and

$$I(\lambda) = \left[-\frac{g(t)\cos\Big((\lambda + \frac{1}{2})t\Big)}{\lambda + \frac{1}{2}}\right]_0^\pi + \frac{1}{\lambda + \frac{1}{2}} \int_0^\pi g'(t)\cos\Big((\lambda + \tfrac{1}{2})t\Big)\, dt.$$

Since g and g' are bounded on $[0, \pi]$, there exists a constant M such that $|g(t)| \le M$ and $|g'(t)| \le M$. Hence

$$|I(\lambda)| \le \frac{2M}{\lambda + \frac{1}{2}} + \frac{1}{\lambda + \frac{1}{2}} \int_0^\pi |g'(t)|\, dt \le \frac{2M + M\pi}{\lambda + \frac{1}{2}}.$$

The right-hand side tends to 0 as $\lambda \to \infty$. Therefore,

$$\boxed{\lim_{\lambda \to \infty} I(\lambda) = 0.}$$

(Equivalently, this also follows from the Riemann–Lebesgue lemma since $g \in L^1([0, \pi])$.)

61. Say that $I = \int_0^\pi \frac{1}{a+b\cos(x)} dx$

$$
\begin{aligned}
I &= \int_0^\pi \frac{1}{a - b\cos x} dx \\
2I &= \int_0^\pi \frac{2a}{a^2 - b^2\cos^2 x} dx \\
I &= a\int_0^\pi \frac{1}{a^2 \sin^2 x + (a^2 - b^2)\cos^2 x} dx. \\
I &= a\int_0^\pi \frac{\csc^2 x}{a^2 + (a^2 - b^2)\cot^2 x} dx \\
&= a\int_{-\infty}^\infty \frac{1}{a^2 + (a^2 - b^2)t^2} dt \\
&= \frac{\pi a}{\sqrt{a^2 - b^2}}.
\end{aligned}
$$

1 Differential Equations